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PROGRESS REPORT NO. 6
ON STUDIES IN THE FIELDS OF
SPACE FLIGHT AND GUIDANCE THEORY

Sponsored by Aero-Astrodynamics Laboratory

NASA

George C. Marshall Space Flight Center, Huntsville, Alabama

NASA-GEORGE C. MARSHALL SPACE FLIGHT CENTER

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ABSTRACT

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This paper contains progress reports of NASA-sponsored studies in the areas of space flight and guidance theory. The studies are carried on by several universities and industrial companies. This progress report covers the period from December 18, 1963 to July 23, 1964. The technical supervisor of the contracts is W. E. Miner, Deputy Chief of the Astrodynamics and Guidance Theory Division, Aero-Astrodynamics Laboratory, Marshall Space Flight Center.

authors first derive some formulas of Keplerian motion involving their six elements, then the perturbation equations, and finally, present the first order solution. It is interesting to observe that no critical angles occur in the second order solution, but that they will appear in a third order solution.

The tenth paper by R. E. Wheeler of Hayes International Corporation presents a statistical procedure for estimating the accuracy that can be expected of a given guidance function. Variations due to changes in launch times, vehicle parameters, and other disturbances are considered. The procedure establishes an upper bound for 2-sigma limits and checks the validity of such limits.

The eleventh paper by R. E. Wheeler of Hayes International Corporation presents the derivation of a mathematical model for fitting the steering function. No end conditions were considered since all constants of integration were combined with unknown constants in the expansion.

The twelfth paper by Daniel E. Dupree, James O'Neil, and Edward Anders of Northeast Louisiana State College presents a method of developing a function $\varphi_{N+1}(\beta')$ previously derived in Progress Report No. 5. The method is detailed in the report and will be implemented here in the near future.

REPUBLIC AVIATION CORPORATION

DIRAC'S GENERALIZED HAMILTONIAN DYNAMICS
AND THE
PONTRYAGIN PRINCIPLE

by

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ACKNOWLEDGMENT

The author would like to express her appreciation for the encouragement provided by W. E. Miner of MSFC in pursuing this investigation. The direction of the effort towards the formulation of a perturbation theory for optimization problems was certainly influenced by discussions with Mr. Miner, who is currently developing such a theory along somewhat different lines.

REPUBLIC AVIATION CORPORATION
Farmingdale, L. I., N. Y.

DIRAC'S GENERALIZED HAMILTONIAN DYNAMICS AND THE PONTRYAGIN PRINCIPLE

by

Mary Payne

SUMMARY

Abst

Dirac's generalized Hamiltonian dynamics is described and applied first to a particular optimization problem and then to a general class of such problems. It is shown that the Dirac formulation leads to a Hamiltonian to which the Pontryagin Maximum Principle can be applied. Further, this Hamiltonian has the property of being canonical in all of its variables, and is thus susceptible to treatment by the methods of classical celestial mechanics. The report closes with a brief discussion of how perturbation techniques, based on the Dirac Hamiltonian, might be developed for the solution of optimization problems.

Author

I. INTRODUCTION

The purpose of this report is to formulate a generalization of the Pontryagin approach for application to optimization problems. This generalization will add nothing new to the basic equations to be solved, but is, rather, intended to lead to perturbation procedures for the solution of these equations. In the Pontryagin formulation of optimization problems a function which bears close resemblance to a Hamiltonian function is introduced. It differs from most classical Hamiltonian functions in two respects: First, the classical Hamiltonian for most problems in dynamics is quadratic in the momenta whereas the Pontryagin Hamiltonian is linear. The second difference is that the Pontryagin Hamiltonian is canonical only in the state variables and their conjugate momenta. In the Pontryagin approach, the control variables are determined, not from Hamilton equations, but by the Pontryagin maximum principle which says that the Hamiltonian must be a maximum in the control variables. The generalization consists in defining a new Hamiltonian, to which the maximum principle can still be applied, but which is canonical in all the variables. The advantage of this new Hamiltonian is that all the methods of classical dynamics now become available for the solution of the problem. In particular, the classical perturbation theories can be applied for obtaining successive closed form approximations for the solution. Most current efforts to solve optimization problems involve numerical integration with the serious defect that initial values of the momenta must be found from an initial set of trial values by some differential correction procedure whose success will in general depend on how close these trial values are to the actual initial conditions.

The construction of the new Hamiltonian is based on a technique developed by Dirac for problems in which the Lagrangian function is linear in the velocities.

It is shown in Section III that the construction of a Hamiltonian for such problems involves special difficulties that are not present in the usual problems of classical dynamics for which the Lagrangian is quadratic in the velocities. Dirac's motivation for this work was his interest in relativistic gravitational fields and quantum electrodynamics. In both problems the Lagrangian is linear in some of the generalized velocities, so that the difficulties that are involved in the construction of a Hamiltonian are identical with those involved in optimization problems. Thus the Dirac formulation, although not originally intended for this purpose, can be applied to optimization problems.

It will be seen that the new Hamiltonian, which will be referred to as the Dirac Hamiltonian, will be linear in all the momenta problems for optimization. This fact makes it very attractive from the point of view of development of a Hamilton-Jacobi perturbation theory since the Hamilton-Jacobi equation will be a linear partial differential equation of first order.

Section II presents some general background material. In Section III, the construction of the Dirac Hamiltonian is discussed in some detail. Section IV presents a development of the Dirac Hamiltonian for a time optimal point-to-point transfer problem. In Section V the connection between the Pontryagin and Dirac Hamiltonians is discussed for the example of Section IV, and in Section V the theory is extended to more general problems. Finally, Section VII presents a brief discussion of the ways in which perturbation procedures might be developed for the solution of optimization problems.

II. BACKGROUND

In the Pontryagin formulation of optimization problems, the variables are classified as state variables x_i which must satisfy certain equations of motion and control variables y_i which appear in the equations of motion:

$$\dot{x}_i = f_i (x,y) \qquad , \qquad i = 1,2,\ldots n. \tag{1}$$

From the state variables x_i and a set of adjoint or conjugate variables ψ_i a Hamiltonian function H_P is constructed which is canonical in the variables x_i and their conjugate momenta ψ_i. That is, the Hamilton equations

$$\dot{x}_i = \frac{\partial H_P}{\partial \psi_i} \quad , \qquad \dot{\psi}_i = \frac{\partial H_P}{\partial x_i} \quad , \tag{2}$$

are satisfied. The Hamiltonian is constructed so that the Hamilton equations for x_i are just the equations of motion, and the equations for ψ_i serve to define the conjugate functions ψ_i. The Hamiltonian H_P is not canonical in the control variables y_i since no momenta conjugate to the y_i appear and hence the $\dot{y}_i$ are not given by partials of H_P with respect to their momenta. The subscript P is used to distinguish the Pontryagin Hamiltonian from a conventional Hamiltonian which is canonical in all of its variables.

For a problem which optimizes x_o with

$$\dot{x}_o = f_o\,(x,y) \ , \tag{3}$$

an additional variable ψ_o is introduced and the Pontryagin Hamiltonian has the form

$$H_P = \sum_{i=0}^{n} \psi_i\, f_i\,(x,y) \ . \tag{4}$$

For a time optimal problem $f_o = 1$, and it is shown (page 20 of Ref. 1) that ψ_o is a negative constant, which may be taken as -1 without loss of generality.

As mentioned above, the Pontryagin Hamiltonian is not canonical in the control variables. The control variables are determined from the Maximum Principle which says that H_P must be a maximum in the control variables if the optimization is a minimization. It is shown in this report that a technique developed by Dirac may be used to define a Hamiltonian H_D which is canonical in all of the variables. This Hamiltonian is usable as a Pontryagin Hamiltonian for application of the Maximum Principle and has the added advantage that the transformation theory of Hamiltonian dynamics is now available for the solution of optimization problems. It is evident from Eq. (4) that H_P is linear in the momenta ψ_i and this property will also hold for the Dirac Hamiltonian H_D, which in fact is linear in all the momenta p_i conjugate to the coordinates q_i, which will be seen to include not only the state and control variables, but also the Lagrange multipliers associated with the Lagrangian formulation of the problem. Thus, for example, the Hamilton-Jacobi equation obtained by substituting

$$p_i = \frac{\partial S(q, \alpha)}{\partial q_i} \tag{5}$$

in H_D will be a linear partial differential equation for the generating function S. Its solution would lead to a canonical transformation, defined by S, to new canonical variables α_i and Q_i obtained from Eq. (5) and the following equation:

$$Q_i = \frac{\partial S(q, \alpha)}{\partial \alpha_i} \tag{6}$$

The Hamiltonian may be written

$$H_D = H_D(\alpha_i) \tag{7}$$

in terms of the new variables, so that

$$\alpha_i = -\frac{\partial H_D}{\partial Q_i} = 0 \qquad \dot{Q}_i = \frac{\partial H_D}{\partial \alpha_i} = \nu_i = \text{constant} \tag{8}$$

or

$$Q_i = \nu_i t + \beta_i . \tag{9}$$

Even if the Hamilton-Jacobi equation is not solvable, the standard perturbation procedures of celestial mechanics would now be available by writing H_D as the sum of H_{D0} and H_{D1} with H_{D0} selected to represent a solvable problem and H_{D1} treated as a perturbation (Ref. 2, pp. 62-74).

In order to obtain the Dirac Hamiltonian H_D, it is necessary to start from a Lagrangian formulation. For a time optimal problem the Lagrangian function is

$$L = 1 + \sum_{i=1}^{n} \lambda_i \left(\dot{x}_i - f_i (x,y) \right) \tag{10}$$

where the λ_i are the usual Lagrange multipliers associated with the equations of motion regarded as differential constraints. To pass from a Lagrangian to a Hamiltonian formulation, one first defines momenta p_i conjugate to the variables q_i (which include the x_i, y_i and λ_i) by the equation

$$p_i = \frac{\partial L}{\partial \dot{q}_i} \tag{11}$$

For the Lagrangian (10), the momenta conjugate to x_i, y_i and λ_i are

$$P_{xi} = \frac{\partial L}{\partial \dot{x}_i} = \lambda_i, \qquad P_{\lambda i} = \frac{\partial L}{\partial \dot{\lambda}_i} = 0\ , \qquad P_{yi} = \frac{\partial L}{\partial \dot{y}_i} = 0 \tag{12}$$

The Hamiltonian is conventionally defined as the function

$$H = \sum_{i=1}^{n} p_i \dot{q}_i - L\ . \tag{13}$$

It is readily shown that this Hamiltonian is a function only of the q's and p's and is independent of the $\dot{q}$'s. This is done by considering the variation in H produced by variations in the q's, $\dot{q}$'s and p's consistent with the defining relations (11) for the p's, but otherwise arbitrary:

$$\begin{aligned} \delta H &= \sum_{i=1}^{n} p_i \,\delta \dot{q}_i + \sum_{i=1}^{n} \dot{q}_i \,\delta p_i - \sum_{i=1}^{n} \frac{\partial L}{\partial q_i} \,\delta q_i - \sum_{i=1}^{n} \frac{\partial L}{\partial \dot{q}_i} \,\delta \dot{q}_i \\ &= \sum_{i=1}^{n} \dot{q}_i \,\delta p_i - \sum_{i=1}^{n} \frac{\partial L}{\partial q_i} \,\delta q_i \end{aligned} \tag{14}$$

The variation in H is independent of the variations in the $\dot{q}$'s and hence it must be possible to write H in such a way that it depends only on q's and p's. In conventional problems in dynamics, Eqs. (11), defining the p's, may be uniquely inverted to give the $\dot{q}$'s as functions of the q's and p's. These expressions for the $\dot{q}$'s may then be substituted for H in the defining Eq. (13) to give a unique expression for H as a function of q's and p's.

For the optimization problem, with the Lagrangian (10), the relations of Eq. (14) still hold, so that the Hamiltonian is still independent of the $\dot{q}$'s. It is, however, no longer unique, as may be seen by direct use of Eqs. (12) and (13):

$$\begin{aligned} H &= \sum_{i=1}^{n} p_i \dot{q}_i - L = \sum_{i=1}^{n} p_{xi} \dot{x}_i + \sum_{i=1}^{n} p_{\lambda i} \dot{\lambda}_i + \sum_{i=1}^{n'} p_{yi} \dot{y}_i - 1 - \sum_{i=1}^{n} \lambda_i (\dot{x}_i - f_i) \\ &= \sum_{i=1}^{n} (p_{xi} - \lambda_i) \dot{x}_i + \sum_{i=1}^{n} p_{\lambda_i} \dot{\lambda}_i + \sum_{i=1}^{n'} p_{yi} \dot{y}_i - 1 + \sum_{i=1}^{n} \lambda_i f_i \\ &= \sum_{i=1}^{n} \lambda_i f_i - 1, \qquad n' = \text{number of control variables} \end{aligned} \tag{15}$$

since the first three sums vanish by virtue of Eqs. (12). One can again make use of Eqs. (12) to write the Hamiltonian as

$$H' = \sum_{i=1}^{n} p_{xi} f_i - 1 \tag{16}$$

which has, of course, the same "value" as H, but has a different functional form. The form (15) would require that all velocities vanish if it is considered as a "true" Hamiltonian, canonical in its variables. The form (16) is substantially the Pontryagin Hamiltonian and is canonical in the state variables.

III. THE DIRAC HAMILTONIAN

In References 3 and 4, Dirac has developed his Hamiltonian formulation for problems in which constraints among the coordinates and momenta are implied by the defining equations for the momenta. The treatment in Reference 3 is more detailed and also more difficult to read than that in Reference 4. Most of the development in Reference 3 is for a Lagrangian homogeneous of the first degree in the velocities. While this restriction involves no loss of generality (the Lagrangian may always be transformed to this form, as shown in Reference 5), it does not appear in Reference 4. The results of the two analyses are substantially the same. The treatment in Reference 4 is in a form more useful for optimization problems. The contents of References 3 and 4 are presented below, for direct application to optimization problems.

The starting point for Dirac's development is a Lagrangian which is a function of N generalized coordinates q_i and their velocities $\dot{q}_i$:

$$L = L(q, \dot{q}) \tag{17}$$

from which momenta p_i conjugate to the coordinates q_i are defined by

$$p_i = \frac{\partial L}{\partial \dot{q}_i} \tag{18}$$

As noted in Section II, if Eqs. (18) may be inverted to give each $\dot{q}_i$ as a unique function of the q's and p's, the classical Hamiltonian development follows. If this is not the case, the classical definition of the Hamiltonian becomes ambiguous, as illustrated by Eqs. (15) and (16). Actually these two equations are special cases of an infinite number of forms for the Hamiltonian:

$$H_1 + \sum_m a_m \varphi_m \tag{19}$$

where H_1 is any form such as in Eqs. (15) or (16), the a_m are arbitrary functions of the q's and p's, and the φ's represent the constraints among the q's and p's implicit in Eq. (18) defining the p's:

$$\varphi_m (q,p) = 0 \tag{20}$$

These constraints may arise because some of the $\dot{q}$'s do not appear in Eqs. (18) or because of redundancy of these equations in the $\dot{q}$'s. Strictly speaking, the expressions (19) cannot really all be regarded as Hamiltonians since by a Hamiltonian one usually means a function of coordinates and their conjugate momenta such that the Hamilton equations

$$\dot{q}_i = \frac{\partial H}{\partial p_i} \qquad \dot{p}_i = - \frac{\partial H}{\partial q_i} \tag{21}$$

are equivalent to the equations of motion of the system described by the Lagrangian L. Thus, the question that Dirac asks is "How may coefficients u_m be chosen from all arbitrary coefficients a_m in Eq. (19) so that, given some H_1 satisfying

$$H_1 = \sum_i p_i \dot{q}_i - L \tag{22}$$

the function

$$H = H_1 + \sum_m u_m \varphi_m \tag{23}$$

is the Hamiltonian for the Lagrangian system L?" As shown in Section II, the function H_1, defined by Eq. (22), may be regarded as a function only of q's and p's. Since the φ_m are also functions only of q's and p's, the function H of Eq. (23) satisfies the first condition for a Hamiltonian, i.e., it is a function only of coordinates q_i and their conjugate momenta p_i. It remains to determine the u_m as functions of the q_i and p_i such that the Hamilton equations describe the motion of the system. It will turn out that the Hamiltonian so obtained is not unique. The essential reason for this is discussed at the end of this section.

It is necessary to make a few remarks about the functions φ_m before proceeding. These functions are assumed to form a complete, independent set of constraints on the q's and p's implied by Eqs. (18). The term "independent" means that no constraint, say φ_k, is implied by the remaining constraints. In this connection, it should be noted that independent constraints $\varphi_k = 0$ and independent functions φ_k are not synonomous terms. The functions q and q^2 are independent but the constraints $q = 0$ and $q^2 = 0$ are not independent; each implies the other. The term "complete" means that every constraint implied by Eqs. (18) is also implied by Eqs. (20) and conversely. It is obvious that the number of constraints M cannot exceed the number of coordinates N. If the Lagrangian is independent of some velocity, say $\dot{q}_k$, it follows that the momentum p_k conjugate to q_k vanishes so that one constraint would be

$$\varphi_1 = p_k = 0 \tag{24}$$

If the Lagrangian is homogeneous of the first degree in the velocities, the momenta will be homogeneous of degree zero in the velocities and hence depend only on the ratio of the velocities. Since there are only N-1 independent ratios of velocities and there are N p's, at least one constraint among the q's and p's must exist. Still another way in which constraints might arise occurs when the velocities $\dot{q}_1$ and $\dot{q}_2$ appear, for example, only in the form $\dot{q}_1 + \dot{q}_2$. Then

$$p_1 = p_2 = \frac{\partial L}{\partial(\dot{q}_1 + \dot{q}_2)} \tag{25}$$

and the corresponding constraint is

$$\varphi = p_1 - p_2 = 0 \tag{26}$$

In the following development the assumptions made on the nature of the constraints is that they be independent, complete, and differentiable. The purpose of this last condition will appear immediately.

It has already been seen (Section II) that the variation in H_1, induced by variations in the q's, $\dot{q}$'s and p's consistent with the defining equations for the momenta, may be written

$$\delta H_1 = \sum_i \dot{q}_i \, \delta p_i - \sum_i \frac{\partial L}{\partial q_i} \, \delta q_i \tag{27}$$

The condition on the variations in the q's, $\dot{q}$'s and p's implies not only that Eq. (18) holds (this was used to cancel out the $\delta \dot{q}$ terms) but that they be such that the induced variations in the φ's shall vanish -- that is that the constraints not be violated. Thus, the following relations among the δq_i and δp_i hold:

$$\delta \varphi_m = \sum_i \frac{\partial \varphi_m}{\partial q_i} \, \delta q_i + \sum_i \frac{\partial \varphi_m}{\partial p_i} \, \delta p_i = 0 \tag{28}$$

These equations may be interpreted as saying that of the 2 N variations, δq_i and δp_i, some M may be determined in terms of the remaining 2 N - M. At this point the meaning of the independence of the φ's may be more precisely stated: the φ's must be such that Eq. (28) form a consistent independent set of linear equations in the δq_i and δp_i.

Recalling that H_1 is a function only of the q's and p's, and using the condition of differentiability on L which implies differentiability of H_1 with respect to its variables, one may write the variation of H_1 in the form

$$\delta H_1 = \sum_i \frac{\partial H_1}{\partial q_i} \, \delta q_i + \sum_i \frac{\partial H_1}{\partial p_i} \, \delta p_i \tag{29}$$

If there were no constraints the δq_i and δp_i could all be regarded as independent and matching coefficients of the δq_i and δp_i in Eqs. (27) and (29) would lead to the usual Hamilton equations. With constraints present, one may proceed as follows: Multiply Eq. (28) by the undetermined multiplier $(-u_m)$ and sum over m, add Eq. (27) and subtract Eq. (29) to obtain

$$\sum_i \left(\dot{q}_i - \frac{\partial H}{\partial p_i} - \sum_m u_m \frac{\partial \varphi_m}{\partial p_i} \right) \delta p_i + \sum_i \left(- \frac{\partial L}{\partial q_i} - \frac{\partial H_1}{\partial p_i} - \sum_m u_m \frac{\partial \varphi_m}{\partial q_i} \right) \delta q_i = 0 \tag{30}$$

Now think of some M of the δq_i and δp_i as being determined in terms of the remaining 2N - M by Eq. (28) and require that the u_m be such that the coefficients of these M variations vanish. The remaining (2N - M) δq_i and δp_i may now be regarded as independent, so that their coefficients must also vanish. Thus all coefficients in Eq. (30) are to vanish and, making use of the Lagrange equations

$$\dot{p}_i = \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} = \frac{\partial L}{\partial q_i} \tag{31}$$

one obtains

$$\begin{aligned} \dot{q}_i &= \frac{\partial H_1}{\partial p_i} + \sum_m u_m \frac{\partial \varphi_m}{\partial p_i} \\ \dot{p}_i &= -\frac{\partial H_1}{\partial q_i} - \sum_m u_m \frac{\partial \varphi_m}{\partial q_i} \end{aligned} \tag{32}$$

Since the φ_m all vanish, it follows that for any variable x

$$\frac{\partial}{\partial x} u_m \varphi_m = u_m \frac{\partial \varphi_m}{\partial x} + \varphi_m \frac{\partial u_m}{\partial m} = u_m \frac{\partial \varphi_m}{\partial x} \tag{33}$$

and hence, defining the Dirac Hamiltonian

$$H_D = H_1 + \sum u_m \varphi_m \tag{34}$$

one may conclude that H_D is a Hamiltonian with Hamilton equations:

$$\dot{q}_i = \frac{\partial H_D}{\partial p_i} \qquad \dot{p}_i = -\frac{\partial H_D}{\partial q_i} \tag{35}$$

The coefficients u_m may be determined as functions of the q's and p's as follows. The equations of motion (35) obtained from the Hamiltonian (30) must be consistent with the constraints (20). This means that not only must the φ_m vanish, but so must their time derivatives. That is, for each m

$$\dot{\varphi}_m = \sum_i \frac{\partial \varphi_m}{\partial q_i} \dot{q}_i + \sum_i \frac{\partial \varphi_m}{\partial p_i} \dot{p}_i$$

$$= \sum_i \frac{\partial \varphi_m}{\partial q_i} \left(\frac{\partial H_1}{\partial p_i} + \sum_m u_m \frac{\partial \varphi_m}{\partial p_i} \right) \tag{36}$$

$$- \sum_i \frac{\partial \varphi_m}{\partial p_i} \left(\frac{\partial H_1}{\partial q_i} + \sum_m u_m \frac{\partial \varphi_m}{\partial q_i} \right)$$

It generally happens that no u's will appear in some of Eqs. (36). In this case, additional constraints among the q's and p's appear, whose time derivatives must also vanish. Those constraints associated with the defining equations for the p's are denoted by φ_m and are called primary constraints. All other constraints are denoted by χ_i and are called secondary constraints. Only the primary constraints appear in the Hamiltonian. All constraints must have vanishing time derivatives, so that secondary constraints arising from $\dot{\varphi}_m = 0$ may lead to additional secondary constraints. This process of equating time derivatives of constraints to zero must be repeated until no further secondary constraints appear. There will then remain a number of equations for the u_m which may be insufficient to determine all M of the u_m. The case in which the remaining equations are insufficient to determine all of the u_m requires special discussion. Any inconsistency in either the constraining equations or the equations for the u_m indicates an original Lagrangian formulation containing inconsistencies.

To see how this process works in detail, it is desirable to introduce the Poisson Bracket notation. If ξ and η are two dynamical variables (functions of q's and p's) their Poisson Bracket (P. B.) is defined by

$$[\xi, \eta] = \sum_i \left\{ \frac{\partial \xi}{\partial q_i} \frac{\partial \eta}{\partial p_i} - \frac{\partial \xi}{\partial p_i} \frac{\partial \eta}{\partial q_i} \right\} \tag{37}$$

from which it follows immediately that

$$[\xi,\eta] = -[\eta,\xi] \qquad [\xi,\xi] = 0$$

$$[\xi,\eta+\zeta] = [\xi,\eta] + [\xi,\zeta] \tag{38}$$

$$[\xi,\eta\zeta] = \eta[\xi,\zeta] + \zeta[\xi,\eta]$$

The usefulness of this notation lies in the following relation:

$$\begin{aligned}\dot{\xi} &= \sum_i \left[\frac{\partial\xi}{\partial q_i}\dot{q}_i + \frac{\partial\xi}{\partial p_i}\dot{p}_i\right] \\ &= \sum_i \left[\frac{\partial\xi}{\partial q_i}\frac{\partial H_D}{\partial p_i} - \frac{\partial\xi}{\partial q_i}\frac{\partial H_D}{\partial q_i}\right] = \left[\xi, H_D\right]\end{aligned} \tag{39}$$

where use has been made of the Hamilton Eqs. (35). Recalling the definition (34) for H_1, one obtains

$$\begin{aligned}\dot{\xi} &= \left[\xi, H_1 + \sum_m u_m\varphi_m\right] \\ &= \left[\xi, H_1\right] + \sum_m u_m\left[\xi, \varphi_m\right] + \sum_m \varphi_m\left[\xi, u_m\right] \\ &= \left[\xi, H_1\right] + \sum_m u_m\left[\xi, \varphi_m\right]\end{aligned} \tag{40}$$

on making use of Eqs. (38).

The condition that a primary constraint have vanishing time derivatives may now be written

$$\dot{\varphi}_i = \left[\varphi_i, H_1\right] + \sum_m u_m \left[\varphi_i, \varphi_m\right] = 0 \tag{41}$$

It may happen that for some φ_k, $[\varphi_k, \varphi_m]$ vanishes for all m, and in this case

$$\left[\varphi_k, H_1\right] = 0 = \chi_1 \tag{42}$$

would appear as a secondary constraint. Secondary constraints could also arise by elimination of u's among some of Eqs. (41). Let the independent secondary constraints obtained from Eq. (41) be denoted by χ_i. It is now required that all $\dot{\chi}_i$ should vanish; that is

$$\dot{\chi}_i = \left[\chi_i, H_1\right] + \sum_m u_m \left[\chi_i, \varphi_m\right] \tag{43}$$

and Eqs. (43) may lead to further secondary constraints. When all the secondary constraints have been found, there will remain a number of independent linear equations in the u_m.

It is now necessary to provide a further classification of the constraints. A constraint is defined as first class if its P. B. with H_1 and with every other constraint vanishes either identically or by virtue of the constraints. All other constraints are second class. Suppose that a set of the primary constraints, denoted by φ_k, is first class. It follows that

$$\left[\varphi_k, H_1\right] = \left[\varphi_k, \varphi_m\right] = \left[\varphi_k, \chi_i\right] = 0 \tag{44}$$

Thus

$$\dot{\varphi}_k = \left[\varphi_k, H_1\right] + \sum_m u_m \left[\varphi_k, \varphi_m\right] = 0$$

$$\dot{\varphi}_{m'} = \left[\varphi_{m'}, H_1\right] + \sum_{m \neq k} u_m \left[\varphi_{m'}, \varphi_m\right] = 0 \tag{45}$$

(cont'd on next page)

$$\dot{\chi}_i = \left[\chi_i, H_1\right] + \sum_{m \neq k} u_m \left[\chi_i, \varphi_m\right] = 0 \qquad (45)\ \text{(cont'd)}$$

and none of the equations requiring time derivatives of the constraints to vanish contain the u_k. Therefore, the u_k are undetermined and the first class constraints appear in the Hamiltonian H with undetermined multipliers. Dirac shows in his paper that the multipliers associated with the second class primary constraints are uniquely determined by those Eqs. (45) corresponding to the second class constraints. The equations corresponding to the first class constraints, whether primary or secondary, yield no information of the u's.

The Dirac Hamiltonian, given by Eq. (34), is now determined in terms of any H_1 consistent with Eq. (22) and the u's determined from Eq. (45). The u's so obtained will, of course, depend on the particular form selected for H_1. The Dirac Hamiltonians obtained from different choices for H_1 may appear, at first glance, to have different forms. This brings up the question, noted at the beginning of this section, of the ambiguity in the Dirac Hamiltonian. It is, of course, immediately obvious that first class primary constraints introduce an ambiguity since their u coefficients are undetermined. There is a further ambiguity which arises from the fact that the Hamiltonian has been constructed to be a function only of q's and p's. Further, the Hamilton equations are satisfied and are such that all constraints are maintained. The validity of the Hamilton equations was obtained from the first order variation of H_1 and the φ's. Now, suppose that some function g (q,p) is such that its first order variation

$$\delta g = \sum_i \frac{\partial g}{\partial q_i} \delta q_i + \sum_i \frac{\partial g_i}{\partial p_i} \delta p_i \qquad (46)$$

vanishes by virtue of the constraints. Such a function is ${\varphi_k}^2$ or $\cos \varphi_k$:

$$\delta\left(\varphi_k^{\,2}\right) = 2\varphi_k\,\delta\varphi_k = 0 \quad \text{since } \varphi_k = 0$$

$$\delta\left(\cos\varphi_k\right) = -\left(\sin\varphi_k\right)\delta\varphi_k = 0 \qquad \text{since} \quad \sin\varphi_k = 0 \tag{47}$$

Since any such function may be added to the Dirac Hamiltonian without changing either the Hamilton equations or the validity of the constraints, an additional ambiguity is introduced besides that inherent in the existence of first class constraints. The Dirac Hamiltonians obtained from different choices of H_1 all lead to the same final equations of motion and all maintain the same constraints. Hence, they must differ only by functions whose first order variation vanishes.

The introduction into H_D of additional terms whose first order variation vanishes has a very practical application: it frequently makes possible the elimination of some of the variables from the Hamiltonian, and reduces the number of equations which must be solved. Just how this works is illustrated in the time optimal orbit transfer problem discussed in Section IV.

IV. THE DIRAC FORMULATION FOR A TIME OPTIMAL TRANSFER PROBLEM

This section illustrates how the Dirac Hamiltonian formulation is applied to optimization problems for the following time optimal transfer problem. For simplicity, the two dimensional problem is chosen. The state variables are the coordinates x and y, their time rates of change ξ and η, and the mass, m. It is assumed that initial and final values of all state variables are specified. The control variables are θ, the direction of thrust, and the rate of fuel flow which is assumed bounded between zero and some fixed upper limit β. Thus, the equations of motion for the problem are:

$$\begin{aligned} \dot{x} &= \xi \qquad \dot{\xi} = -\frac{\partial V}{\partial x} + \frac{c\beta\cos^2\alpha}{m}\sin\theta \\ \dot{y} &= \eta \qquad \dot{\eta} = -\frac{\partial V}{\partial y} + \frac{c\beta\cos^2\alpha}{m}\cos\theta \\ \dot{m} &= -\beta\cos^2\alpha \end{aligned} \tag{48}$$

where the thrust is, of course, $-c\dot{m}$, and the constraint on the fuel flow is carried by the variable α. Forces other than thrust acting on the vehicle are assumed derivable from a potential function $V(x,y)$ dependent only on position of the vehicle. The transfer time is to be minimized, subject to the equations of motion (48), which are to be regarded as differential constraints. Since the Dirac formulation can give information only on first order variations in the time integral of the Lagrangian, no information on the nature of the extremals for this integral appears in this section. In the next section the Maximum Principle is incorporated in the theory, and discussions of the nature of the solution obtained in this section are thus deferred. Introducing Lagrange multipliers, the Lagrangian for this optimization problem is

$$\begin{aligned} L = 1 &+ \lambda_1 (\dot{x} - \xi) + \lambda_2 (\dot{y} - \eta) \\ &+ \lambda_3 \left(\dot{\xi} + \frac{\partial V}{\partial x} - \frac{c \beta \cos^2 \alpha}{m} \sin \theta \right) \\ &+ \lambda_4 \left(\dot{\eta} + \frac{\partial V}{\partial y} - \frac{c \beta \cos^2 \alpha}{m} \cos \theta \right) \\ &+ \lambda_5 (\dot{m} + \beta \cos^2 \alpha) \end{aligned} \tag{49}$$

The Lagrangian L contains, explicitly, the differential constraints and the bounding constraints on $\dot{m}$. It does not, however, contain the constraints on the initial and final values of the state variables. This omission means that the constants of integration from the Hamilton equations must be ultimately used to determine initial values for the control variables and the Lagrange multipliers. It will be seen later that this represents a serious defect in the theory, and that an effort should be made to find a Lagrangian formulation which explicitly includes all constraints on the problem to be solved.

In the Lagrangian (49) the state variables x, y, ξ, η and m, the control variables θ and α, and the Lagrange multipliers λ_i will all be regarded as coordinates. The only velocities appearing are those corresponding to the state variables. The momenta conjugate to the coordinates are obtained by differentiation of the Lagrangian with respect to the corresponding velocities:

$$
\begin{aligned}
&p_x = \lambda_1 \qquad && p_{\lambda_i} = 0 \\
&p_y = \lambda_2 && p_\theta = 0 \\
&p_\xi = \lambda_3 && p_\varphi = 0 \\
&p_\eta = \lambda_4 && \\
&p_m = \lambda_5 &&
\end{aligned}
\tag{50}
$$

No velocities appear in the defining equations for the momenta and thus all of these equations represent primary constraints. Further, all of the constraints are independent. The constraints are labeled as follows:

$$
\begin{aligned}
&\varphi_1 = p_x - \lambda_1 = 0 \qquad && \varphi_6 = p_{\lambda_1} = 0 \qquad && \varphi_{11} = p_\theta = 0 \\
&\varphi_2 = p_y - \lambda_2 = 0 && \varphi_7 = p_{\lambda_2} = 0 && \varphi_{12} = p_\alpha = 0 \\
&\varphi_3 = p_\xi - \lambda_3 = 0 && \varphi_8 = p_{\lambda_3} = 0 && \\
&\varphi_4 = p_\eta - \lambda_4 = 0 && \varphi_9 = p_{\lambda_4} = 0 && \\
&\varphi_5 = p_m - \lambda_5 = 0 && \varphi_{10} = p_{\lambda_5} = 0 &&
\end{aligned}
\tag{51}
$$

The function H_1 is selected to be

$$
H_1 = \lambda_1 \xi + \lambda_2 \eta - \lambda_3 \left(\frac{\partial V}{\partial x} - \frac{c\beta \cos^2 \alpha}{m} \sin \theta \right) \tag{52}
$$

$$
- \lambda_4 \left(\frac{\partial V}{\partial y} - \frac{c\beta \cos^2 \alpha}{m} \cos \theta \right) - \lambda_5 \beta \cos^2 \alpha - 1
$$

which is consistent with Eq. (22).

To obtain the expressions for the φ's, it is necessary to obtain the P.B.'s of the φ's among themselves and of each φ with H_1. The P.B.'s of the φ's among themselves are

$$[\varphi_1, \varphi_6] = -[\varphi_6, \varphi_1] = -1$$
$$[\varphi_2, \varphi_7] = -[\varphi_7, \varphi_2] = -1$$
$$[\varphi_3, \varphi_8] = -[\varphi_8, \varphi_3] = -1 \qquad \text{all other } [\varphi_i, \varphi_j] = 0 \qquad (53)$$
$$[\varphi_4, \varphi_9] = -[\varphi_9, \varphi_4] = -1$$
$$[\varphi_5, \varphi_{10}] = -[\varphi_{10}, \varphi_5] = -1$$

and the P. B's of the φ's with H_1 are

$$[\varphi_1, H_1] = \lambda_3 \frac{\partial^2 V}{\partial x^2} + \lambda_4 \frac{\partial^2 V}{\partial x \partial y} \qquad [\varphi_6, H_1] = -\xi$$

$$[\varphi_2, H_1] = \lambda_3 \frac{\partial^2 V}{\partial x \partial y} + \lambda_4 \frac{\partial^2 V}{\partial y^2} \qquad [\varphi_7, H_1] = -\eta$$

$$[\varphi_3, H_1] = -\lambda_1 \qquad [\varphi_8, H_1] = \frac{\partial V}{\partial x} - \frac{c\beta \cos^2 \alpha}{m} \sin \theta$$

$$[\varphi_4, H_1] = -\lambda_2 \qquad [\varphi_9, H_1] = \frac{\partial V}{\partial y} - \frac{c\beta \cos^2 \alpha}{m} \cos \theta$$

$$[\varphi_5, H_1] \doteq \frac{c\beta \cos^2 \alpha}{m^2} (\lambda_3 \sin\theta + \lambda_4 \cos\theta) \qquad [\varphi_{10}, H_1] = \beta \cos^2 \alpha$$

$$[\varphi_{11}, H_1] = -\frac{c\beta \cos^2 \alpha}{m} (\lambda_3 \cos \theta - \lambda_4 \sin \theta)$$

$$[\varphi_{12}, H_1] = \beta \sin 2\alpha \left\{ \frac{c}{m} (\lambda_3 \sin \theta + \lambda_4 \cos \theta) - \lambda_5 \right\} \qquad (54)$$

The time derivatives of the φ's are obtained by making use of Eq. (40) and they must be equated to zero:

$$\dot{\varphi}_1 = - u_6 + \lambda_3 \frac{\partial^2 V}{\partial x^2} + \lambda_4 \frac{\partial^2 V}{\partial x \partial y} = 0$$

$$\dot{\varphi}_2 = - u_7 + \lambda_3 \frac{\partial^2 V}{\partial x \partial y} + \lambda_4 \frac{\partial^2 V}{\partial y^2} = 0$$

$$\dot{\varphi}_3 = - u_8 - \lambda_1 = 0$$

$$\dot{\varphi}_4 = - u_9 - \lambda_2 = 0$$

$$\dot{\varphi}_5 = - u_{10} + \frac{c\beta \cos^2\alpha}{m^2} (\lambda_3 \sin \theta + \lambda_4 \cos \theta) = 0 \qquad (55)$$

$$\dot{\varphi}_6 = \quad u_1 - \xi = 0$$

$$\dot{\varphi}_7 = u_2 - \eta = 0$$

$$\dot{\varphi}_8 = u_3 + \frac{\partial V}{\partial x} - \frac{c\beta \cos^2 \alpha}{m} \sin \theta = 0$$

$$\dot{\varphi}_9 = u_4 + \frac{\partial V}{\partial y} - \frac{c\beta \cos^2 \alpha}{m} \cos \theta = 0$$

$$\dot{\varphi}_{10} = u_5 + \beta \cos^2 \alpha = 0$$

$$\dot{\varphi}_{11} = - \frac{c\beta \cos^2 \alpha}{m} (\lambda_3 \cos \theta - \lambda_4 \sin \theta) = 0$$

$$\dot{\varphi}_{12} = \beta \sin 2\alpha \left\{ \frac{c}{m} (\lambda_3 \sin \theta + \lambda_4 \cos \theta) - \lambda_5 \right\} = 0$$

It will be noted that the first ten φ's give immediately the first ten u's. No u's occur in the last two and hence the requirement that $\dot{\varphi}_{11}$ and $\dot{\varphi}_{12}$ vanish leads to two secondary constraints:

$$\chi_1 = \cos^2 \alpha\,(\lambda_3 \cos\theta - \lambda_4 \sin\theta) = 0$$

$$\chi_2 = \sin 2\alpha \left\{\frac{c}{m}(\lambda_3 \sin\theta + \lambda_4 \cos\theta) - \lambda_5\right\} = 0 \tag{56}$$

where the factors c, β and m, known to be nonvanishing, have been omitted. These secondary constraints are, in a way, somewhat embarrassing since they both appear as products, so that further discussion requires consideration of the various combinations in which the factors may vanish. The occurrence of this problem is, however, not surprising; it is just the way in which the "switching function" in the conventional theory would first appear. To completely specify the "switching function" requires consideration of second variations to distinguish minima from other stationary values of the time integral of the Lagrangian. There is no provision for this in the Dirac theory, and further discussion of this point will be deferred. First, the Dirac Hamiltonian is obtained and in the next section the way in which the Maximum Principle complements the Dirac theory is discussed.

The ways in which the vanishing of the χ's may be guaranteed are:

Case 1. $\cos\alpha = 0$

Case 2. $\sin\alpha = 0$, $\lambda_3 \cos\theta - \lambda_4 \sin\theta = 0$

Case 3. $\lambda_3 = \lambda_4 = \lambda_5 = 0$

Case 4. $\lambda_3 = \lambda_4 = \sin\alpha = 0$

Case 5. $\lambda_3 \cos\theta - \lambda_4 \sin\theta = 0$, $\frac{c}{m}(\lambda_3 \sin\theta + \lambda_4 \cos\theta) - \lambda_5 = 0$

For a complete analysis of this time optimization problem, each of these possibilities should be examined in detail with recognition of the fact that the nature of the problem may require the use of different Hamiltonians for different portions of the final optimum trajectory. Since, however, the purpose in this report is merely to illustrate the application of the Dirac technique to optimization problems, only the first two possibilities are discussed. These correspond to the conventional solution of the problem by the Pontryagin principle. It might be mentioned that the occurrence of possibilities 3, 4, and 5

$$+ p_{\lambda 1}\left(\lambda_3 \frac{\partial^2 V}{\partial x^2} + \lambda_4 \frac{\partial^2 V}{\partial x \partial y}\right) + p_{\lambda 2}\left(\lambda_3 \frac{\partial^2 V}{\partial x \partial y} + \lambda_4 \frac{\partial^2 V}{\partial y^2}\right)$$

$$- \lambda_1 p_{\lambda 3} - \lambda_2 p_{\lambda 4} + \frac{c\,\beta \cos^2 \alpha}{m^2} (\lambda_3 \sin\theta + \lambda_4 \cos\theta)\, p_{\lambda 5}$$

$$+ \frac{-\lambda_1 \cos\theta + \lambda_2 \sin\theta}{\lambda_3 \sin\theta + \lambda_4 \cos\theta}\, p_\theta \qquad (64) \quad \text{(cont'd)}$$

The form of this Hamiltonian differs from that of case 1 only in the p_θ term.

It will be recalled that it was stated in Section IV that the Dirac Hamiltonian is not unique and that terms whose first variation vanishes identically may be added at will. One way in which differing Dirac Hamiltonians could be obtained would be to start with the λ's in H_1 replaced by the momenta conjugate to the state variables, which is consistent with the first five primary constraints. Had this been done, the resulting Dirac Hamiltonians (59) and (64) for cases 1 and 2 would have p_x, p_y, p_ξ and p_η instead of λ_1, λ_2, λ_3 and λ_4, respectively. It is a relatively easy matter to show that the difference between these Hamiltonians does indeed have vanishing first order variation. Consider, for example, the difference D_1 between the $p_{\lambda 1}$ terms:

$$D_1 = p_{\lambda 1}\left\{(\lambda_3 - p_\xi) \frac{\partial^2 V}{\partial x^2} + (\lambda_4 - p_\eta) \frac{\partial^2 V}{\partial x \partial y}\right\} \qquad (65)$$

for which the variation is

$$\delta D_1 = \delta p_{\lambda 1}\left\{(\lambda_3 - p_\xi) \frac{\partial^2 V}{\partial x^2} + (\lambda_4 - p_\eta) \frac{\partial^2 V}{\partial x \partial y}\right\}$$

$$+ p_{\lambda 1} \quad \left\{ (\delta\lambda_3 - \delta p_\xi) \frac{\partial^2 V}{\partial x^2} + (\delta\lambda_4 - \delta p_\eta) \frac{\partial^2 V}{\partial x \partial y}\right. \qquad (66)$$

$$\left. + (\lambda_3 - p_\xi)\left(\delta \frac{\partial^2 V}{D x^2}\right) + (\lambda_4 - p_\eta)\left(\delta \frac{\partial^2 V}{\partial x \partial y}\right)\right\}$$

The constraints φ_3 and φ_4 guarantee that the first bracket vanishes and the constraint φ_6 guarantees that the $p_{\lambda 1}$ term vanishes independent of the variations in $p_{\lambda 1}$, λ_3, p_ξ, x and y. The remaining terms in the difference of the two H_D's are treated similarly. Thus, the H_D obtained is essentially independent of whichever of the two forms outlined above is selected for H_1.

The fact that any term of vanishing first variation can be added to H_D without changing its essential character may now be used to transform the Hamiltonians (59) and (64) into the same form. This is achieved by eliminating the variables θ and p_θ. It is readily verified that one of the functions

$$g_{\substack{1\\2}} = p_\xi \sin\theta + p_\eta \cos\theta \mp \sqrt{p_\xi^2 + p_\eta^2} \tag{67}$$

vanishes for case 2 as a consequence of the $\bar{\chi}_2$ constraint. Further, the variation in g is given by

$$\delta g_{\substack{1\\2}} = \left(\sin\theta \mp \frac{p_\xi}{\sqrt{p_\xi^2 + p_\eta^2}}\right)\delta p_\xi + \left(\cos\theta \mp \frac{p_\eta}{\sqrt{p_\xi^2 + p_\eta^2}}\right)\delta p_\eta$$
$$+ (p_\xi \cos\theta - p_\eta \sin\theta)\,\delta\theta \tag{68}$$

and again from the $\bar{\chi}_2$ constraint the coefficients of δp_ξ, δp_η and $\delta\theta$ vanish. Finally, since any function f multiplied by g will also have vanishing first order variation, it follows that $p_\xi \sin\theta + p_\eta \cos\theta$ may be replaced by $\pm\sqrt{p_\xi^2 + p_\eta^2}$ in the Dirac Hamiltonian (64). Since $\dot{\theta}$ and hence θ are undetermined by the Hamiltonian (59) for case 1, the same substitution may also be made there. The Hamiltonians now differ only in their p_θ terms, and since the dependence on θ has been essentially replaced by p_ξ and p_η these terms may be omitted without loss of generality.

Anticipating the results of application of the Maximum Principle, it may be noted that for case 2, it will be required that $p_\xi \sin\theta + p_\eta \cos\theta$ must be positive. Using this condition, one obtains the Dirac Hamiltonian as

$$H_D = p_x \xi + p_y \eta - p_\xi \frac{\partial V}{\partial x} - p_\eta \frac{\partial V}{\partial y} + \frac{c\beta \cos^2 \alpha}{m} \sqrt{p_\xi^{\,2} + p_\eta^{\,2}}$$

$$- p_m \beta \cos^2 \alpha - 1 + p_{\lambda 1} \left(p_\xi \frac{\partial^2 V}{\partial x^2} + p_\eta \frac{\partial^2 V}{\partial x \partial y} \right) \tag{69}$$

$$+ p_{\lambda 2} \left(p_\xi \frac{\partial^2 V}{\partial x \partial y} + p_\eta \frac{\partial^2 V}{\partial y^2} \right) - p_x p_{\lambda 3} - p_y p_{\lambda 4}$$

$$+ \frac{c\beta \cos^2 \alpha}{m^2} \sqrt{p_\xi^{\,2} + p_\eta^{\,2}}\, p_{\lambda 5}$$

and, finally, at this stage the terms in $p_{\lambda i}$ may be omitted in the same way as the p_θ. All of the essential information is carried by the state variables, their momenta, and the control variable α with the Hamiltonian

$$H_D = p_x \xi + p_y \eta - p_\xi \frac{\partial V}{\partial x} - p_\eta \frac{\partial V}{\partial y}$$

$$+ \frac{c\beta \cos^2 \alpha}{m} \sqrt{p_\xi^{\,2} + p_\eta^{\,2}} - p_m \beta \cos^2 \alpha - 1 \tag{70}$$

which is canonical in all the variables. This is a very compact form for the Hamiltonian. It has, however, one disadvantage. The momenta p_ξ and p_η enter irrationally. There may, therefore, be some advantage in retaining the dependence on θ, together with the two forms (59) and (64) for the Hamiltonians corresponding to cases 1 and 2, respectively.

V. INCORPORATION OF THE MAXIMUM PRINCIPLE IN THE DIRAC FORMULATION

The Dirac Hamiltonian obtained for the time optimal problem described in Section IV was written in a number of different forms. It was noted that the terms in the momenta conjugate to the state variables were just the Pontryagin Hamiltonian

$$H_P = p_x \xi + p_y \eta - p_\xi \left(\frac{\partial V}{\partial x} - \frac{c \beta \cos^2 \alpha}{m} \sin \theta\right)$$
$$- p_\eta \left(\frac{\partial V}{\partial y} - \frac{c \beta \cos^2 \alpha}{m} \cos \theta\right) - p_m \beta \cos^2 \alpha \tag{71}$$

so that corresponding to Eqs. (59) and (64)

$$H_D = H_P - 1 + p_{\lambda 1}\left(\lambda_3 \frac{\partial^2 V}{\partial x^2} + \lambda_4 \frac{\partial^2 V}{\partial x \partial y}\right)$$
$$+ p_{\lambda 2}\left(\lambda_3 \frac{\partial^2 V}{\partial x \partial y} + \lambda_4 \frac{\partial^2 V}{\partial y^2}\right) - \lambda_1 p_{\lambda 3} - \lambda_2 p_{\lambda 4} \tag{72}$$
$$+ \frac{c \beta \cos^2 \alpha}{m^2}\left(\lambda_3 \sin \theta + \lambda_4 \cos \theta\right) p_{\lambda 5} + u_{11} p_\theta$$

with

u_{11} undetermined for case 1

$$u_{11} = \frac{- \lambda_1 \cos \theta + \lambda_2 \sin \theta}{\lambda_3 \sin \theta + \lambda_4 \cos \theta} \quad \text{for case 2} \tag{73}$$

Now the Pontryagin principle requires that H_P be maximized with respect to the control variables. Since the only way in which H_P and H_D differ in their dependence on the control variables is in the $p_{\lambda 5}$ term in H_D, and since $p_{\lambda 5}$ vanishes, maximization of H_P with respect to the control variables implies the corresponding maximization of H_D and conversely. The first condition for maximization is that

$$\frac{\partial H_P}{\partial \alpha} = \frac{\partial H_D}{\partial \alpha} = 0$$
$$\frac{\partial H_P}{\partial \theta} = \frac{\partial H_D}{\partial \theta} = 0 \tag{74}$$

These conditions are guaranteed for H_D which has been so constructed that the Hamilton equations will yield vanishing time derivatives for p_α and p_θ, the momenta conjugate to α and θ. It was these conditions which led to the secondary constraints with five cases to be considered. Only the first two cases, corresponding to the conventional Pontryagin formulation of the problem, have been analysed in detail.

In the conventional treatment, the bounds on $\dot{m}$ are not explicitly written into the Lagrangian. To obtain the conventional Pontryagin Hamiltonian, one could just omit the $\cos^2 \alpha$ factors in Eq. (71) and apply later the condition that the fuel flow, represented by β has lower bound zero and upper bound, say, β_{max}. Thus, the conventional Pontryagin Hamiltonian can be written as

$$H_P = p_x \xi + p_y \eta - p_\xi \frac{\partial V}{\partial x} - p_\eta \frac{\partial V}{\partial y} + \frac{c\beta}{m}\left(p_\xi \sin\theta + p_\eta \cos\theta\right) - p_m \beta \qquad (75)$$

with

$$0 \le \beta \le \beta_{max} \qquad (76)$$

In this form H_P varies linearly with β and hence the maximum of H_P with respect to β will be on one of the bounds, and which bound is to be used will be determined by the sign of the switching function

$$k = \frac{c}{m}\left(p_\xi \sin\theta + p_\eta \cos\theta\right) - p_m \qquad (77)$$

according to the criterion that

$$\begin{aligned} \beta &= 0 & k &\le 0 \\ \beta &= \beta_{max} & k &\ge 0 \end{aligned} \qquad (78)$$

The maximization with respect to θ requires that

$$\frac{\partial H_P}{\partial \theta} = \frac{c\beta}{m}\left(p_\xi \cos\theta - p_\eta \sin\theta\right) = 0$$

$$\frac{\partial^2 H}{\partial \theta^2} = -\frac{c\beta}{m}\left(p_\xi \sin\theta + p_\eta \cos\theta\right) \leq 0 \tag{79}$$

The first of these conditions implies that

$$p_\xi \sin\theta + p_\eta \cos\theta = \pm\sqrt{p_\xi^{\,2} + p_\eta^{\,2}} \tag{80}$$

and the second requires that the + sign be used in Eq. (80) for $\beta \neq 0$.

It will be noted that the Dirac formulation with the bounds on $\dot{m}$ included in the Lagrangian requires (for cases 1 and 2) that the bounds of the fuel flow be used and that Eq. (80) hold. The selection of the positive sign in Eq. (80) and the operation of the switching function according to Eq. (78) are the essential additional information obtained from the Maximum Principle. It should be mentioned that if the bounds on $\dot{m}$ were explicitly included in the Pontryagin formulation (i. e., by writing the constraint on $\dot{m}$ as $\beta \cos^2 \alpha$) the same five cases for investigation would appear as for the Dirac theory.

The analysis of this time optimal transfer problem has shown that the Dirac formulation can be used instead of the Pontryagin formulation and that the Maximum Principle can be applied to the Dirac Hamiltonian. It is shown in Section VI that these conclusions can be extended to a general class of optimization problems.

VI. THE DIRAC FORMULATION FOR A CLASS OF OPTIMIZATION PROBLEMS

The construction of the Dirac Hamiltonian for application to more general optimization problems is not difficult to carry out. Suppose, for example, that the optimization problem is to minimize the time integral of a function $f_0(x, y)$ where x represents the state variables $x_1, x_2, \ldots, x_N$ subject to the differential constraints

$$\dot{x}_i = f_i\,(x,y) \tag{81}$$

and y represents the control variables $y_1, y_2, \ldots, y_K$. It will be assumed that any bounded control variables are replaced by an expression of the form

$$y_{min}\cos^2\alpha + y_{max}\sin^2\alpha \tag{82}$$

where y_{min} and y_{max} are the bounds on the control variable. A similar form will be employed for any bounded state variable with the differential equations suitably rewritten in terms of the parameter α. Thus, it may be assumed that the state and control variables are all unbounded.

Introducing Lagrange multipliers, the Lagrangian for the optimization is

$$L = f_o\,(x,y) + \sum_{i=1}^{N} \lambda_i \left(\dot{x}_i - f_i\,(x,y)\right) \tag{83}$$

with coordinates $x_1, x_2, \ldots, x_N, y_1, y_2, \ldots, y_K, \lambda_1, \lambda_2, \ldots \lambda_N$. The momenta conjugate to these coordinates are

$$\begin{aligned} &p_{xi} = \lambda_i && i = 1,2,3,\ldots,N \\ &p_{\lambda i} = 0 && \\ &p_{yk} = 0 && k = 1,2,\ldots,K \end{aligned} \tag{84}$$

It is convenient to write the corresponding primary constraints in the form

$$\begin{aligned} &\varphi_i = p_{xi} - \lambda_i = 0 \\ &\psi_i = p_{\lambda i} = 0 \\ &\omega_k = p_{yk} = 0 \end{aligned} \tag{85}$$

As before, the function H_1 is defined by

$$H_1 = \sum_i p_i \dot{q}_i - L = \sum_{i=1}^{N} \lambda_i f_i - f_0 \tag{86}$$

and the Dirac Hamiltonian is given by

$$H_D = H_1 + \sum_{i=1}^{N} u_i \varphi_i + \sum_{i=1}^{N} v_i \psi_i + \sum_{k=1}^{K} w_k \omega_k \tag{87}$$

where the u's, v's and w's must be suitably determined from the requirement that the time derivatives of all primary and secondary constraints must vanish. To obtain the time derivatives of the primary constraints, use is made of their P.B.'s among themselves and with H_1:

$$\begin{aligned}
&\left[\varphi_i, \omega_k\right] = \left[\psi_i, \omega_k\right] = \left[\varphi_i, \varphi_j\right] = \left[\psi_i, \psi_j\right] = \left[\omega_k, \omega_\ell\right] = 0 \\
&\left[\varphi_i, \psi_j\right] = -\left[\psi_j, \varphi_i\right] = -\delta_{ij} \\
&\left[\varphi_i, H_1\right] = -\sum_j \lambda_j \frac{\partial f_j}{\partial x_i} + \frac{\partial f_0}{\partial x_i} \\
&\left[\psi_i, H_1\right] = -f_i \\
&\left[\omega_k, H_1\right] = -\sum_j \lambda_j \frac{\partial f_j}{\partial y_k} + \frac{\partial f_0}{\partial y_k}
\end{aligned} \tag{88}$$

from which one readily obtains

$$\begin{aligned}
&\sum_{j=1}^{N} u_j \left[\varphi_i, \varphi_j\right] + \sum_{j=1}^{N} v_j \left[\varphi_i, \psi_j\right] + \sum_{k=1}^{K} w_k \left[\varphi_i, \omega_k\right] = -v_i \\
&\sum_{j=1}^{N} u_j \left[\psi_i, \varphi_j\right] + \sum_{j=1}^{N} v_j \left[\psi_i, \psi_j\right] + \sum_{k=1}^{K} w_k \left[\psi_i, \omega_k\right] = u_i
\end{aligned} \tag{89}$$

(cont'd on next page)

$$\sum_{j=1}^{N} u_j \left[\omega_\ell, \varphi_j\right] + \sum_{j=1}^{N} v_j \left[\omega_\ell, \psi_j\right] + \sum_{k=1}^{K} w_k \left[\omega_\ell, \omega_k\right] = 0 \qquad (89)\ \text{(cont'd)}$$

so that

$$\dot{\varphi}_i = - v_i - \sum_j \lambda_j \frac{\partial f_j}{\partial x_i} + \frac{\partial f_0}{\partial x_i} = 0$$

$$\dot{\psi}_i = u_i - f_i = 0 \qquad (90)$$

$$\dot{\omega}_\ell = - \sum_j \lambda_j \frac{\partial f_j}{\partial y_\ell} + \frac{\partial f_0}{\partial y_\ell} = 0$$

From the $\dot{\varphi}$ and $\dot{\psi}$ equations one obtains the u's and v's:

$$v_i = - \sum_j \lambda_j \frac{\partial f_j}{\partial x_i} + \frac{\partial f_0}{\partial x_i}$$

$$(91)$$

$$u_i = f_i$$

The $\dot{\omega}$ equations do not contain any of the undetermined multipliers u_i, v_i, w_k and hence are secondary constraints χ_ℓ:

$$\chi_\ell = - \sum_j \lambda_j \frac{\partial f_j}{\partial y_\ell} + \frac{\partial f_0}{\partial y_\ell} = 0 \qquad (92)$$

whose P.B.'s are:

$$\left[\chi_\ell, \varphi_i\right] = - \sum_j \lambda_j \frac{\partial^2 f_j}{\partial x_i \partial y_\ell} + \frac{\partial^2 f_0}{\partial x_i \partial y_\ell}$$

$$(93)$$

$$\left[\chi_\ell, \psi_i\right] = - \frac{\partial f_i}{\partial y_\ell}$$

(cont'd on next page)

$$[\chi_\ell, \omega_k] = -\sum_j \lambda_j \frac{\partial^2 f_j}{\partial y_k \partial y_\ell} + \frac{\partial^2 f_0}{\partial y_\ell \partial y_k}$$

(93) (cont'd)

$$[\chi_\ell, H_1] = 0$$

so that

$$\chi_\ell = \sum_{i=1}^{N} u_i \left(-\sum_{j=1}^{N} \lambda_j \frac{\partial^2 f_j}{\partial x_i \partial y_\ell} + \frac{\partial^2 f_0}{\partial x_i \partial y_\ell} \right)$$

$$- \sum_{i=1}^{N} v_i \frac{\partial f_i}{\partial y_\ell} + \sum_{k=1}^{K} w_k \left[-\sum_{j=1}^{N} \lambda_i \frac{\partial^2 f_j}{\partial y_k \partial y_\ell} + \frac{\partial^2 f_0}{\partial y_\ell \partial y_k} \right] \tag{94}$$

These equations may or may not lead to further secondary constraints depending on the $\frac{\partial^2 f_j}{\partial y_k \partial y_\ell}$. At any rate, completion of the calculation of the w_k and determination of the existence of first class constraints is a routine matter for any particular problem. The Dirac Hamiltonian becomes, on using the expressions for the u's, v's, φ's, ψ's and ω's

$$H_D = \sum_{j=1}^{N} \lambda_j f_j - f_0 + \sum_{j=1}^{N} u_j \varphi_j + \sum_{j=1}^{N} v_j \psi_j + \sum_k w_k \omega_k$$

$$= \sum_{j=1}^{N} \lambda_j f_j - f_0 + \sum_{j=1}^{N} f_j (p_{xj} - \lambda_j)$$

$$- \sum_{j=1}^{N} \left[\left(\sum_{i=1}^{N} \lambda_i \frac{\partial f_i}{\partial x_j} \right) - \frac{\partial f_0}{\partial x_j} \right] p_{\lambda j} + \sum_k w_k p_{yk} \tag{95}$$

$$= \sum_{j=1}^{N} p_{xj} f_j - f_0 - \sum_{i,j=1}^{N} \lambda_i p_{\lambda j} \frac{\partial f_i}{\partial x_j} + \sum_{j=1}^{N} p_{\lambda j} \frac{\partial f_0}{\partial x_j} + \sum_k w_k p_{yk}$$

$$= H_P + \text{terms linear in } p_{\lambda j} \text{ and } p_{yk}$$

where some of the w_k may vanish and others may be indeterminate, indicating the presence of first class constraints. The function H_P is

$$H_P = -f_0 + \sum_{j=1}^{N} p_{xj} f_j \tag{96}$$

which is consistent with the Pontryagin formulation.

This Dirac Hamiltonian may be used in place of the Pontryagin Hamiltonian in the Maximum Principle, since any contribution of the terms in $p_{\lambda j}$ and p_{yk} in the application of this principle will contain $p_{\lambda j}$ or p_{yk} as vanishing coefficients.

VII. HAMILTONIAN TECHNIQUES FOR THE SOLUTION OF OPTIMIZATION PROBLEMS

In the preceding sections a Hamiltonian formulation for optimization problems has been developed. It has been applied to a particular optimization problem and it has been seen that the Maximum Principle can be incorporated in the formulation. Further, it has been shown that this formulation can be generalized for other optimization problems. In this section a perturbation theory for the solution of optimization problems is outlined. First, however, one comment should be made on a defect of the method.

This defect is that the constraints on the initial and final values of the state variables have not been explicitly incorporated in the formulation. Just how this might be done is far from clear. It may, however, be noted that incorporation of the bounds on fuel flow leads to secondary constraints which imply that the fuel flow operates on its bounds for cases 1 and 2 without recourse to the Maximum Principle. Explicit inclusion of constraints on the initial and final values of the state variables might lead to additional secondary constraints on the control variables which would automatically fit the final solution of the Hamilton equations to initial and final values. It will be recalled that, in addition to cases 1 and 2, which have been discussed in some detail, cases 3, 4, and 5 may occur. These cases probably correspond, in some sense, to singular solutions of the problem which are significant only for particular sets of initial and final values. Their treatment and interpretation would be greatly clarified if the initial and final values were made an integral part of the formulation.

It should be mentioned that the theory developed in this report assumes that a complete set of initial and final values has been imposed on the state variables. No difficulty is anticipated in relaxation of this limitation. Incorporation of transversality conditions into the Dirac formulation appears to be straightforward. This would, of course, have to be done for application of the theory to orbit transfer problems.

The theory as developed in the preceding sections is in a form particularly suitable for the Hamilton-Jacobi approach. The Hamilton-Jacobi equation derived from the Hamiltonian H_D in the forms (59) and (64) would be a linear first order partial differential equation. Neither of these equations separates. One could, however, undertake a perturbation procedure and write

$$H_D = H_{D0} + H_{D1} \tag{97}$$

with H_{D0} selected to represent a solvable problem. The selection of H_{D0} would depend on the particular problem to be solved. In general, one undertakes to split H_D so that not only is the H_{D0} problem solvable, but also that H_{D1} is, in some sense, small compared with H_{D0}. It would also be desirable to choose H_{D0} in such a way that its Hamilton-Jacobi equation is separable. It is not easy to satisfy all of these conditions on H_{D0}, as will be seen from the examples discussed below. Considerable further analysis is necessary before a satisfactory perturbation theory for optimization problems can be worked out in detail. Two ways in which the theory might be applied are:

Low Thrust Problems

For such problems it is assumed that the maximum thrust is small compared with the gravitational forces acting on the vehicle. In addition, some of the gravitational forces might be small in comparison with others. Thus, H_{D1} might be chosen to include all terms involving β (since if the thrust is small, β is small) as well as those terms involving the small gravitational forces. Then H_{D0} would represent the optimal trajectory for a vehicle moving under a gravitational force derivable from a potential V_0. If the potential V_0 is just the two body potential then H_{D0} represents the classical Kepler problem in a rather unconventional form. For the problem discussed in Section IV, for instance, there would be many more variables than are normally associated with the two body problem because of the presence of the p's. Further, the Hamilton-Jacobi equation associated with H_{D0} does not separate for this case. Since, however, the solution of the two body problem is well known, it should be possible to somehow construct a solution of the Hamilton-Jacobi equation which could be used as a basis for a perturbation theory for the low thrust problem.*

<u>High Thrust Problems</u>

In this case one could select H_{D1} to include all terms involving V since the gravitational forces would be assumed small compared to the thrust. The Hamiltonian H_{D0} would then represent the optimal trajectory for a vehicle with no forces other than thrust. The associated Hamilton-Jacobi equation does not separate for this case either. As in the low thrust problems, however, the solution for the H_{D0} can be obtained in closed form and is available for use in the same way as the Kepler problem for the low thrust case.

It thus appears that the development of a Hamiltonian perturbation theory for optimization problems is feasible. Further work in this area is planned, and results will be submitted as they are obtained.

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* Such a solution is currently under investigation along somewhat different lines by W. E. Miner of MSFC.

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INVESTIGATION OF SUFFICIENCY CONDITIONS AND THE HAMILTON JACOBI APPROACH TO OPTIMAL CONTROL PROBLEMS

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TABLE OF CONTENTS

INTRODUCTION AND SUMMARY

This is the final report on contract NAS 8-11020 entitled "Optimum Trajectory Study".

In this section we will try to give a verbal account of the problems considered, the reasons for considering them, and the main results obtained. The remaining sections, while having independent introductions, will contain the mathematical analysis.

The major objective of this study was to examine the use of Hamilton Jacobi partial differential equations in determining fields of optimum trajectories and to study sufficiency conditions. Since a great number of optimal control problems can, with a slight reformulation, be posed as time optimal problems, our attention is focused throughout on problems of this type.

If given initial data, say time $t = t_o$, state $x = x_o$ for a time optimal problem, the reachable set (in Euclidean (n+1) dimensional time-state space) is defined to be the set of all points (t, x) with time $t \geq t_o$ and state x such that it can be attained in time t by a trajectory of the dynamical system with an admissible control. Under very mild conditions on the dynamical system equations and the control set, it is known that a time optimal point to point transfer will lead to a trajectory which lies on the boundary of the reachable set. Conversely, trajectories which lie on the boundary of the reachable set are excellent candidates for being time optimal for some point to point transfer, and thus conditions which single them out are of interest. Now a point is on the boundary of the reachable set if in every neighborhood of it there are points not in the reachable set; i.e., points not attainable by trajectories of the dynamical system. This leads one naturally to notions of controllability.

Following the definition of Kalman, a linear system is said to be completely controllable at time t_o if every state can be attained (with $\mathcal{L}_2$ control) in finite time by a trajectory of the system having arbitrary initial data (t_o, x_o). Thus one can examine whether the terminal data has been chosen so that the mission is possible. It is of further interest to define local controllability, i.e., a system is locally controllable along a solution trajectory $\varphi(t)$ if for some $t_1 > t_o$ all points in some state space neighborhood of $\varphi(t_1)$ are attainable in time t_1 by trajectories with admissible controls. Obviously trajectories along which a system is locally controllable cannot remain on the boundary of the reachable set, and hence this becomes a test for optimality. It might also be remarked that while for linear systems one could expect global controllability results, for nonlinear systems it is natural to expect only local results.

In Section I, the Kalman criterion for complete controllability for a linear system is derived in a simple manner (corollary I.1) and an extension is obtained for a special form of nonlinear system (Theorem I.2).

In Section II, the nonlinear system $\dot{x}(t) = g(t, x(t))+H(t,x(t))u(t)$, x an n vector, H an nxr matrix, u and r vector valued control with $1 \leq r < n$, is studied. If $B(t, x)$ is an $(n-r)xn$ matrix, of maximal rank, such that $B(t,x)H(t,x) \equiv 0$, the local controllability of the above system is shown to be closely related to the integrability of the pfaffian system $B(t,x)dx - B(t,x)\ g(t, x)dt = 0$. In particular, the above nonlinear system is defined to be completely controllable if the associated pfaffian system is not integrable. Theorem II.1 then shows that in the special case of a linear system, this definition yields a criterion for complete controllability equivalent to that of Kalman. This new criterion is useful since it does not depend on the knowledge of a fundamental solution matrix for a time varying linear system. Its use is demonstrated by obtaining the result that an n dimensional system, formed from a single n^{th} order linear time varying differential equation of the form $x^{(n)}(t) + a_1(t)\ x^{(n-1)}(t) + \ldots + a_n(t)\ x(t) = u(t)$, is completely controllable. (Here u is a scalar valued control). This result was previously known if the functions $a_i(t)$ were constant.

The remainder of section II deals with local controllability in a neighborhood of singular arcs. It is shown that local tests, which depend on examining the controllability of the variational equation along a singular arc will always be non-conclusive. Along an optimal singular arc the system is truly not locally controllable, however it is shown by example (example II.2) that singular arcs can exist along which the system is locally controllable. These can be thought of as inflection points in function space, of the functional (time) which is to be extremized. They are analogous to inflection points which arise when extremizing a real valued function F on a manifold in Euclidean space; i.e., non-extremal points at which the map F induces on the tangent space of the manifold into the tangent space of the reals, vanishes.

These arcs are singular also in the sense of the classical calculus of variations, hence the Hilbert differentiability condition fails to hold along them, and classical sufficiency conditions fail.

In section III, the study of feedback control via the Pontriagin maximum principle and Hamilton Jacobi theory is begun. Often the feedback control which the maximum principle prescribes, is discontinuous in the state variables, which in turn leads to a Hamilton Jacobi equation with discontinuous coefficients. This is impractical both from a theoretical and computational viewpoint. The first part of section III deals mainly with the reason for this discontinuity, and yields conditions such that the maximum principle would prescribe a continuous or even C^1 (once continuously differentiable) control. Theorems III.4 and III.5 then show that whenever a control problem merely satisfies the conditions of Fillipov for the existence of an optimal control, there exists an approximate problem (the precise definition of this precedes theorem III.4) for which the maximum principle gives a C^1 control, and such that for any given $\epsilon > 0$, an optimal trajectory of the original problem will be in an ϵ neighborhood of that for the approximate problem.

The remainder of section III deals with the Hamilton Jacobi theory for these smooth approximate problems, and for the special case of the control appearing linearly, an easy construction for the approximating problem is shown, while an example (example III.1) is worked out in detail to demonstrate the results.

Two sets of references are given, the first for sections I and II, the second for section III.

CONTROLLABILITY AND THE SINGULAR PROBLEM

INTRODUCTION TO SECTIONS I AND II

The concept of complete controllability of linear systems was introduced by R. E. Kalman [1]. It is part of the purpose of this paper to extend the concept to nonlinear systems, with control appearing linearly. All systems considered are of this form.

Geometrically, a linear system is completely controllable at time t_o if any state can be attained in finite time by a trajectory of the system having arbitrary initial data x_o at time t_o. The motivation for the extension of this concept to nonlinear systems came largely from results obtained in [2] and from the geometric interpretation of non-integrability of pfaffians given in [3] and [4]. In particular, Caratheodory gives an argument to show that if, for a single pfaffian equation, there are points in every neighborhood of a given point which are not "reachable" from the given point by curves satisfying the equation, the equation is integrable. This result was generalized to systems of pfaffians in [4]. There is a difficulty in applying these ideas to pfaffian systems which are quite naturally associated with control systems having control appearing linearly. (See § II.) The reason for this is that usually the independent variable t appears explicitly in the pfaffian system, hence its integral curves, which can be related back to solutions of the control system, and are used to connect neighboring points to a given point, must have t parametrized as $t(\sigma)$, a monotone function of σ. This is not the case in the proofs in [3] and [4], and with this restriction, in general the results of these papers are no longer valid.

The relation between singular problems and controllability arises quite naturally from the pfaffian approach and can be anticipated from results obtained by LaSalle in [5]. In § II we define the concept of a totally singular arc, i.e., an arc satisfying the differential constraining equations, for which there exists an adjoint vector such that the maximum principle yields no information as to the optimality of any of the components of the control along this arc. In particular, if the system were linear and admitted no totally singular arc, the system would be proper in the sense of LaSalle [5] and completely controllable in the sense of Kalman [6]. Even if the controls are merely restricted to be $\mathcal{L}_2$ (Lebesgue square integrable) functions, it is shown that totally singular arcs can exist and comprise some or all of the boundary of the attainable set, thereby being optimal trajectories for certain time optimal control problems. These are also precisely the arcs along which the system need not be locally controllable, i.e., if we assume initial data x_o given at time t_o, there may exist points in every state space neighborhood of a point $\varphi^v(t_1)$ of a totally singular arc φ^v, which are not attainable in time $t_1 > t_o$ by trajectories of the system with $\mathcal{L}_2$ controls. Here φ^v denotes the solution of the system with control v. Precisely, if for every $t_1 > t_o$ there exist points in every state space neighborhood of $\varphi^v(t_1)$, which are not attainable with $\mathscr{L}_2$ control in time t_1, the arc φ^v is totally singular. However it is shown by example that there do exist totally singular arcs about which the system is locally controllable.

§I. COMPLETE CONTROLLABILITY FOR LINEAR AND MILDLY NONLINEAR SYSTEMS

Throughout this section H will denote an nxr matrix valued function of t, which is in $\mathscr{L}_2\left[t_o, t_1\right]$ for any given finite $t_1 > t_o$. Controls will be $\mathscr{L}_2$, vector valued functions, We begin with the following basic Lemma.

Lemma I.1 A necessary and sufficient condition that there exist an rxn matrix valued function V(t) in $\mathscr{L}_2\left[t_o, t_1\right]$, such that for some $t_1 > t_o$, $\int_{t_o}^{t_1} H(\tau)V(\tau)d\tau$ is non-singular, is that for some $t_1 > t_o$ $\int_{t_o}^{t_1} H(\tau)H^T(\tau)d\tau$ is non-singular.

Proof Sufficiency is immediate by choosing $V(\tau) = H^T(\tau)$. To show necessity assume there exist V, $t_1 > t_o$, such that $\int_{t_o}^{t_1} H(\tau)V(\tau)d\tau$ is non-singular, but $\int_{t_o}^{\bar{t}} H(\tau)H^T(\tau)d\tau$ is singular for all $\bar{t} > t_o$, in particular $\bar{t} = t_1$. This implies there exists a constant vector $c \neq 0$ such that $c\left(\int_{t_o}^{t_1} H(\tau)H^T(\tau)d\tau\right)c^T = 0$, and since $H(\tau)H^T(\tau)$ is positive semi-definite, we obtain $cH(t) = 0$ almost everywhere in $\left[t_o, t_1\right]$. Thus

$\int_{t_o}^{t_1} H(\tau)V(\tau)d\tau = 0$ which contradicts the non-singularity of

$\int_{t_o}^{t_1} H(\tau)V(\tau)d\tau$. ∎

We next consider the system

$$\text{(1-1)} \qquad \dot{x}(t) = H(t)u(t) \;,\quad x(t_o) = x_o, \quad u \in \mathcal{L}_2\left[t_o, t_1\right].$$

Define

$$M(t_o, t_1) \equiv \int_{t_o}^{t_1} H(\tau)H^T(\tau)d\tau \;.$$

<u>Theorem I.1</u> A necessary and sufficient condition for the system (1-1) to be completely controllable at t_o is that there exists $t_1 > t_o$ such that $M(t_o, t_1)$ is non-singular.

<u>Proof</u>: (Sufficiency) Let $\overline{x}$ be any given point in E^n, Euclidean n space. We will show $\overline{x}$ is attainable from x_o at time t_1. Indeed pick $u(t) = H^T(t)\xi$, $\xi \in E^n$. We desire $\overline{x} = x(t_1) = x(t_o) + \left(\int_{t_o}^{t_1} H(\tau)H^T(\tau)d\tau\right)\xi$ or $\xi = M^{-1}(t_o,t_1)\,(\overline{x} - x(t_o))$.

(Necessity). Assume $M(t_o, t_1)$ is singular for all $t_1 > t_o$. This implies (see proof of lemma I.1) that there exists a constant vector $c \neq 0$ such that $c\,H(t) \equiv 0$ p.p. Since x_o is arbitrary, let it be such that $c \cdot x_o = 0$. We will show the point c is not attainable from x_o. Indeed suppose for some u and t_1, $c = x_o + \int_{t_o}^{t_1} H(\tau)u(\tau)d\tau$. Then

$c \cdot c = \|c\|^2 = c \cdot x_0 + c\int_{t_0}^{t_1} H(\tau)u(\tau)d\tau = 0$, a contradiction to the fact that $c \neq 0$. ■

Corollary I.1 (Kalman) The linear system

$$(1\text{-}2) \qquad \dot{x}(t) = A(t)x(t) + H(t)u(t), \qquad x(t_0) = x_0$$

is completely controllable at t_0 if and only if

$\int_{t_0}^{t_1} \Phi(t_0,\tau)H(\tau)H^T(\tau)\Phi^T(t_0,\tau)d\tau$ is non-singular for some $t_1 > t_0$.

Here $\Phi(t,\tau)$ denotes a fundamental solution of the homogeneous system $\dot{x}(t) = A(t)\,x(t)$.

Proof: Make the transformation $y(t) = \Phi^{-1}(t, t_0)\,x(t)$. Then x satisfies (1-2) if an only if y satisfies

$$(1\text{-}3) \qquad \dot{y}(t) = \Phi(t_0,t)\,H(t)u(t), \quad y(t_0) = x_0.$$

(Note $\Phi(t_0, t) = \Phi^{-1}(t, t_0)$.) From the transformation, it follows that the system (1-2) is completely controllable if and only if the system (1-3) is completely controllable, i.e., from theorem I.1 that there exists a $t_1 > t_0$ such that

$\int_{t_0}^{t_1} \Phi(t_c,\tau)\,H(\tau)\,H^T(\tau)\,\Phi^T(t_0,\tau)d\tau$ is non-singular. ■

Some special results for nonlinear systems

We next consider the nonlinear system

$$(1\text{-}4) \qquad \mathring{x}(t) = g(t, x(t)) + H(t)u(t), \qquad x(t_o) = x_o$$

with the assumptions: i) $\left| g^j(t,x) \right| \leq M$, $j = 1, 2, \ldots, n$. ii) $\left| g^j(t,x) - g^j(t, \bar{x}) \right| \leq m \left\| x - \bar{x} \right\|$, $j = 1, 2, \ldots, n$. iii) g is continuous as a function of t for each x.

Again let $M(t_o, t_1) = \int_{t_o}^{t_1} H(\tau)H^T(\tau)d\tau$.

Theorem I.2 A sufficient condition that the set of points attainable by trajectories of the system (1-4) with $\mathcal{L}_2$ control be all of E^n is that $M(t_o, t_1)$ be non-singular for some $t_1 > t_o$.

Remark Rather than state the theorem in this manner, one might consider merely saying that the system (1-4) is completely controllable at t_o. However, this notion has not been defined for nonlinear systems, and it does not seem reasonable to this author to define it in such a global fashion for these systems.

Proof For arbitrary u, (1-4) has a solution designated φ^u which satisfies

$$(1\text{-}5) \quad \varphi^u(t) \equiv x_o + \int_{t_o}^{t} g(\tau, \varphi^u(\tau))d\tau + \int_{t_o}^{t} H(\tau)u(\tau)d\tau .$$

Let $\bar{x}$ be any given point in E^n. We desire a control such that for some point finite $t_1 > t_o$, $\varphi^u(t_1) = \bar{x}$. It suffices to consider controls which come from a finite dimensional subspace of $\mathcal{L}_2$, in particular the controls considered will be of the form $u(t) = H^T(t)\xi$ where $\xi \in E^n$. Hence the notation φ^ξ rather than φ^u will be used.

Define a mapping $\mathcal{F}: E^n \rightarrow E^n$ as follows:

Let $\alpha(\xi) \equiv \int_{t_o}^{t_1} g(\tau, \varphi^\xi(\tau))d\tau$, and define

$\mathcal{F}(\xi) \equiv M^{-1}(t_o, t_1)\left[\bar{x} - \alpha(\xi) - x_o\right]$. From (1-5) it follows that a fixed point of $\mathcal{F}$ will yield a value ξ such that $\varphi^\xi(t_1) = \bar{x}$.

It is well known that with the conditions imposed on g [7, th. 7.4 - Chapter I], φ^ξ is a continuous function of ξ in the topology $C[t_o, t_1]$, i.e., the topology induced by the supremum norm. Thus $\alpha(\xi)$ is a continuous function of ξ, and $\mathcal{F}$ is a continuous function of ξ.

We next show that there exists a K such that $\|\xi\| \leq K \rightarrow \|\mathcal{F}(\xi)\| \leq K$. Letting $\|\xi\| = \sum_{i=1}^{n} |\xi_i|$, and $\|M^{-1}\|$ be any matrix norm, since $|g^j| \leq M$, for any ξ, $\|\alpha(\xi)\| \leq n(t_1 - t_o)M$:

Letting $K \equiv \|M^{-1}(t_o, t_1)\| \left[\|\bar{x}\| + nM(t_1 - t_o) + \|x_o\|\right]$, it follows that for any ξ, $\|\mathcal{F}(\xi)\| \leq K$, hence in particular $\mathcal{F}$ maps the ball $\{\xi \in E^n: \|\xi\| \leq K\}$ continuously into itself. Thus $\mathcal{F}$ has a fixed point. ∎

<u>Remark</u> The result obtained in this theorem is not surprising in view of theorem (I.1) and the boundedness condition on the vector g. Also the condition $M(t_o, t_1)$ non-singular for some $t_1 > t_o$ is much stronger than

it need be. For example, if we consider a linear system of the form (1-2) and $H(t)$ is a column vector with one component zero, then $M(t_o, t_1)$ is singular for all $t_1 \geq t_o$, yet the system can certainly be completely controllable.

§ II. NONLINEAR SYSTEMS WITH LINEAR CONTROL; THE SINGULAR PROBLEM

In this section, we consider extending the notion of complete controllability to systems of the form

$$\text{(2-1)} \qquad \dot{x}(t) = g(t, x(t)) + H(t, x(t))u(t)$$

where g is an n-vector, H an $n \times r$ matrix, while u is an $\mathcal{L}_2$ control vector. It is assumed that g and H are C^1 in all arguments. Throughout, the stipulation $1 \leq r < n$ is required to hold.

Let $B(t, x)$ be a C^1, $(n-r) \times n$ matrix with rank (n-rank H) at each point (t, x) in some domain $\mathcal{D}$ of interest, such that

$$\text{(2-2)} \qquad B(t,x)\, H(t, x) \equiv 0 , \qquad (t, x) \in \mathcal{D} .$$

Since $r < n$, we know that rank $B \geq 1$ for all (t, x).

With the system (2-1), associate the pfaffian system

$$\text{(2-3)} \qquad B(t, x)dx - B(t, x)\, g(t, x)dt = 0.$$

Let b be an arbitrary linear combination of the rows b^{ν} of B, taken with C^1 scalar valued coefficients $\alpha_{\nu}(t, x)$, i.e.,

$b(t, x) = \sum_{\nu} \alpha_{\nu}(t, x)\, b^{\nu}(t, x)$. Throughout, b will be used to denote such a linear combination which is <u>not</u> identically zero.

<u>Definition II.1</u> The pfaffian system (2-3) is integrable at the point $(\bar{t}, \bar{x})$ if there exists a C^1 scalar valued function $\psi(t, x)$ and an $\epsilon > 0$ such that for some b,

$$\psi_x(t, x) = b(t, x), \quad \psi_t(t, x) = -b(t, x) \cdot g(t, x)$$

for $\bar{t} \leq t < \bar{t} + \epsilon$, $|x - \bar{x}| < \epsilon$.

Essentially this states that for some b,

(2-4) $b(t, x)dx - b(t, x) \cdot g(t, x)dt$

is an exact differential in a "neighborhood" of $(\bar{t}, \bar{x})$. It should be noted that any integrating factor can be included in the coefficients of the linear combination of the rows b^{ν}.

The notion of integrability of a pfaffian system is, of course, related to the property of completencess of an associated system of partial differential equations. To show the relation, let $C(x)$, $x \in E^n$, be a smooth $(n-r)$xn matrix, and $K(x)$ a smooth nxr matrix, both of maximum rank, such that $C(x)K(x) \equiv 0$. With the pfaffian system

(2-4) $C(x)dx = 0$

we associate the system of partial differential equations $K^T(x) \dfrac{\partial f(x)}{\partial x} = 0$.

Each row k^i of K^T can be considered as defining a vector field X^i which locally generates a one parameter semi group of diffeomorphisms, $\{T_i(t)\}$, see for example [8, p. 10]. In turn, such a semi group determines a vector field. If for each i, j = 1, 2, ..., r and for all arbitrarily small fixed $\mathcal{T}$, the vector field determined by $\{T_j(\mathcal{T})\ T_i(t)\ T_j(-\mathcal{T})\}$ is linearly dependent on the fields X^i, the system of partial differential equations is said to be complete. If it is not complete, the number m of linearly independent fields formed in this manner is called the index of both the pfaffian system and the associated partial differential equation system [4].

From the results in [4], it easily follows that <u>the pfaffian system (2-4) is integrable (definition II.1) if and only if the index m is such that $m+r < n$</u>. If the index m is such that $m+r = n$, Chow [4] shows that there is a neighborhood of a point $x_o \in E^n$ such that all points in this neighborhood are attainable by curves satisfying (2-5). From the viewpoint of local controllability for a control system, we can interpret this as follows. <u>If the pfaffian system associated with the control system</u>

$$(2\text{-}5) \qquad \dot{x}(t) = K(x(t))u(t)\ , \qquad\qquad x(t_o) = x_o$$

<u>has index m, where K is a continuous nxr matrix function of $x \in E^n$ with constant rank r, and $m+r = n$, then every point in some neighborhood of x_o is attainable by trajectories of (2-5) with measurable controls</u>. Indeed, since all points in some neighborhood of x_o are attainable by

absolutely continuous curves satisfying $C(x,(t))\ \dot{x}(t) = 0$ almost everywhere, we must only show that such a curve also satisfies (2-5) for some control u. But $C(x(t))\ \dot{x}(t) = 0 \Longrightarrow \dot{x}(t)$ is a linear combination of the columns of $K(x(t))$, since $CK \equiv 0$. Thus there exists $u(t)$ such that $\dot{x}(t) = K(x(t))u(t)$ for almost all t. Since K has rank r, it has a continuous left inverse on its range, from which it follows that u is measurable.

Before stating an explicit criterion for complete controllability of a system of the form (2-1) one may ask: <u>What should one expect the definition to yield</u>? This can presently be answered as follows. Since the definition should extend that given for a linear system of the form (1-2) which is a special case of (2-1), one expects:

a) If $g(t, x) \equiv A(t)x$, $H(t, x) \equiv H(t)$, then the criterion which defines complete controllability at t_o for (II.1) should be equivalent with the condition $\int_{t_o}^{t_1} \Phi(t_o, t)H(t)H^T(t)\,\Phi^T(t_o, t)dt$ non-singular for some $t_1 > t_o$, as given in corollary I.1.

b) There should be a geometric interpretation of the condition, e.g., what points are attainable from the initial point in finite time? In the linear system there were global attainability results, i.e., any point could be attained from the initial point via a trajectory of the system. In the nonlinear problem, one would expect at most local results of this nature.

The approach will be to state a criterion for complete controllability of (2-1) which we will show satisfies a). We then use this criterion to try to establish a geometric interpretation as mentioned in b). Of course, how the definition of complete controllability should be extended is somewhat a matter of personal opinion.

<u>Definition II.2</u> The system (2-1) is completely controllable at $(\bar{t}, \bar{x}) \in \mathcal{D}$ if the associated pfaffian system (2-2) is not integrable at $(\bar{t}, \bar{x})$.

It will next be shown that this criterion is equivalent to the condition given in corollary I.1 for the special case of the linear system (1-2). In this case it suffices to take $B = B(t)$ in forming the pfaffian system equivalent to (2-3). Also, in taking the linear combination of the rows of B to form the single pfaffian as in (2-4), we can consider the scalar functions α_ν as function of only t. Indeed we must only show that if the pfaffian form

$$(2\text{-}6) \qquad b(t)dx - b(t)\, A(t)x\, dt$$

has an integrating factor, then this integrating factor, denoted by μ, can be taken as a function of only t. To obtain this, suppose $\bar{\mu}(t, x)$ is such that $\bar{\mu}(t, x)\, b(t)dx - \bar{\mu}(t, x)b(t)A(t)x\, dt$ is an exact differential. Then $\bar{\mu}_{x_j}\, b^i - \bar{u}_{x_i}\, b^j = 0$ for all $i, j = 1, 2, \ldots, n$, and $\bar{\mu}_t b + \bar{\mu}\dot{b} = -\bar{\mu}_x\, b\, A\, x - \bar{\mu}\, b\, A$. Define $\mu(t) = \bar{\mu}(t, 0)$, noting that for the linear system $\mathcal{D} = (t_o, \infty) \times E^n$ which implies $(t, 0) \in \mathcal{D}$ for $t > t_o$. It follows that $\mu(t)$ is also an integrating factor.

Since it is sufficient to consider both μ and the α_ν as functions of only t, there is no loss of generality in considering that if the pfaffian system

(2-7) $\quad B(t)dx - B(t)\,A(t)x\,dt = 0$

associated with (1-2) is integrable, then (2-6) is an exact differential.

Since x appears linearly, definition II.1 simplifies for such systems, and is: <u>The pfaffian system (2-7) is integrable at the point $\bar{t}$ if there exists a C^1 scalar valued function $\Psi(t, x)$ and an $\epsilon > 0$ such that for some b,</u>

$$\Psi_x(t, x) = b(t), \quad \Psi_t(t, x) = -b(t)\,A(t)\,x$$

<u>for $\bar{t} \leq t < \bar{t} + \epsilon$</u>. (Note: Under the assumptions on B and H, Ψ_{xt} and Ψ_{tx} exist and are equal).

Define:

$$W(t_o, t_1) = \int_{t_o}^{t_1} \Phi(t_o, t)H(t)H^T(t)\,\Phi^T(t_o, t)dt\ .$$

Then corollary I.1 states that the system (1-2) is completely controllable at t_o if and only if there exists a $t_1 > t_o$ such that $W(t_o, t_1)$ is non-singular.

<u>Remark 1</u>. If A and H are constant matrices, Kalman [1] shows that this condition is equivalent to the condition: rank $\left[A, AH, \ldots A^{n-1}H\right] = n$.

Remark 2. While the above condition given for the constant coefficient case can be directly checked, $W(t_o, t_1)$ depends on knowledge of a fundamental solution $\Phi(t, t_o)$ which is not always easily obtainable.

Remark 3. It is easily seen that $W(t_o, t_1)$ is a positive semi-definite matrix. Thus if $W(t_o,t_1)$ is non-singular, $W(t_o, t)$ is non-singular for all $t \geq t_1$.

The main purpose of this section will be to show that the condition II.2 for complete controllability of (1-2) is equivalent to $W(t_o, t_1)$ being non-singular for some $t_1 > t_o$. This condition has the advantage of not depending on knowledge of a fundamental solution.

Before stating the main theorem, a simple computation yields, for $t_o < t_1 < t_2$,

$$W(t_o, t_2) = W(t_o, t_1) + \Phi(t_o, t_1)\, W(t_1, t_2)\, \Phi^T(t_o, t_1).$$

Thus if $W(t_1, t_2)$ is non-singular (positive definitive) it follows that $W(t_o, t_2)$ is also non-singular (positive definite). The reverse implication need not be true.

Theorem II.1 A necessary and sufficient condition that $W(t_1, t_2)$ be non-singular for all $t_2 > t_1$ is that the pfaffian (2-7) be not integrable at t_1.

For ease in both using and proving this theorem, we list the implications and their contrapositives.

I.A <u>Necessary condition</u>: $W(t_1, t_2)$ non-singular for all $t_2 > t_1$

$\Longrightarrow$ pfaffian (2-7) is not integrable at t_1.

I.B <u>Necessary; contrapositive</u>: Pfaffian (2-7) integrable at $t_1 \Longrightarrow$

$W(t_1, t_2)$ is singular for some $t_2 > t_1$.

I.C <u>Sufficient condition</u>: Pfaffian (2-7) not integrable at t_1

$\Longrightarrow W(t_1, t_2)$ is non-singular for all $t_2 > t_1$.

I.D <u>Sufficient; contrapositive</u>: $W(t_1, t_2)$ singular for some $t_2 > t_1$

$\Longrightarrow$ pfaffian (2-7) is integrable at t_1.

<u>Proof</u>: (We shall prove I.B and I.D)

Assume the pfaffian (2-7) is integrable at t_1. Then there is a vector b, which is a linear combination of the rows of B, and an $\epsilon > 0$ such that $\dot{b}(t) = -b(t)A(t)$, for $t_1 \leq t \leq t_1 + \epsilon$. Let $\Phi(t, t_1)$; $\Phi(t_1, t_1) = I$, be the fundamental solution of $\dot{x} = A(t)\, x$. Then the vector b admits the representation $b(t) = c\, \Phi^{-1}(t, t_1) = c\, \Phi(t_1, t)$ for some constant vector c. Let $h(t)$ be any column of $H(t)$. Then $0 = b(t)h(t) = c\, \Phi(t_1, t)\, h(t)$. Since h was an arbitrary column of H, and W is positive semi-definite, we have $c\, W(t_1, t)\, c^T = 0$ for $t_1 \leq t \leq t_1 + \epsilon$ showing that there exists a $t_2 > t_1$ such that $W(t_1, t_2)$ is singular.

Assume, next, that $W(t_1, t_2)$ is singular for some $t_2 > t_1$. From remark 3, it follows that $W(t_1, t)$ is singular for all $t_1 \leq t \leq t_2$. This implies there exists a vector $c(t_2)$ such that $c(t_2)W(t_1, t_2)c^T(t_2) = 0$.

Since the integrand of the integral defining $W(t_1, t_2)$ is continuous,

$$c(t_2)\,\Phi(t_1, t)H(t)H^T(t)\,\Phi^T(t_1, t)c^T(t_2) \equiv 0 \quad \text{for } t_1 \leq t \leq t_2.$$

It follows that $0 \equiv c(t_2)\,\Phi(t_1, t)H(t) \equiv c(t_2)\,\Phi^{-1}(t, t_1)H(t)$, thus b defined by $b(t) \equiv c(t_2)\,\Phi^{-1}(t, t_1)$ is an admissible vector in the sense that $b(t)\,H(t) \equiv 0$, i.e., b lies in the subspace spanned by the rows of B.

Define the scalar valued function $\Psi(t,x) = c(t_2)\,\Phi^{-1}(t, t_1)x$. Then $\Psi_x(t, x) = b(t)$, $\Psi_t(t, x) = -b(t)\,A(t)x$ for $t_1 \leq t \leq t_2$ showing that the pfaffain (2-7) is integrable at t_1. $\blacksquare$

The following illustrates the advantage of a definition of complete controllability for linear systems which does not depend on knowledge of a fundamental solution.

It is known that an n dimensional system which is formed from a single n^{th} order equation having constant coefficients and the control as forcing term is completely controllable. We next show that this is also true for time varying systems of the form

$$x^{(n)}(t) + a_1(t)\,x^{(n-1)}(t) + \ldots + a_n(t)\,x(t) = u(t).$$

Specifically we shall show that for any t_o, the associated pfaffian is not integrable implying $W(t_o, t_1)$ is non-singular for all $t_1 > t_o$.

We take the equivalent linear system of the form

$\dot{y}(t) = A(t)\,y(t) + h(t)\,u(t)$ where

$$A(t) = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ \vdots & & & \ddots & 0 \\ 0 & & \cdots & 0 & 1 \\ -a_n, & -a_{n-1}, & \cdots, & & -a_1 \end{bmatrix}, \qquad h(t) = \begin{bmatrix} 0 \\ \vdots \\ \vdots \\ 0 \\ 1 \end{bmatrix}.$$

One can choose $B(t)$ as the $(n-1) \times n$ matrix

$$B(t) = \begin{bmatrix} 1 & 0 & 0 & \dots & 0 & 0 \\ 0 & 1 & 0 & \dots & 0 & 0 \\ \vdots & & & & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 1 & 0 \end{bmatrix} .$$

The pfaffian system equivalent to (2-7) is then

$$\begin{aligned} dx_1 - x_2\, dt &= 0 \\ dx_2 - x_3\, dt &= 0 \\ \vdots \qquad & \vdots \\ dx_{n-1} - x_n dt &= 0. \end{aligned} \tag{2-8}$$

If (2-8) were to be integrable there must exist scalar valued functions $\alpha_j(t)$, not all zero, so that the single pfaffian

$$\sum_{j=1}^{n-1} \alpha_j(t)\, dx_j + 0\, dx_n - \sum_{j=1}^{n-1} \alpha_j(t)\, x_{j+1}\, dt$$

is an exact differential. But this would imply $\alpha_j(t) = 0$, $j = 1, 2, \dots, (n-1)$, which shows (2-8) is not integrable for any t_0.

Geometric Interpretation, Local Controllability, and the Singular Problem

By associated a pfaffian system of the form (2-3) with the system (2-1), it is conspicuous that the stress is taken away from the functional form of the elements of the matrix H, and placed only on what the range of H(t, x), considered as an operator on E^r, is. This obviously should be the case when controls are required to be only $\mathcal{L}_2$ functions.

In [9], Markus and Lee consider a system of the form $\dot{x} = f(x, u)$, $f \in C^1$ in $E^n \times \Omega$, where Ω a compact set contained in E^r with 0 in its interior, is the range set of the control. Assuming $f(0, 0) = 0$ and letting $A = f_x(0, 0)$, $H = f_u(0, 0)$, it is shown that if the linear system $\dot{x} = Ax + Hu$ is completely controllable, then the set of points from which the origin can be reached in finite time by trajectories of $\dot{x} = f(x, u)$, is an open connected set containing the origin. Kalman [10] pointed out that a similar result can be obtained for a system of the form $\dot{x} = f(t, x, u)$ by assuming the linear approximation is completely controllable in terms of the criterion given in corollary 1.1.

The system

$$\text{(2-9)} \qquad \dot{x}(t) = f(t, x(t), u(t)) \;, \qquad\qquad x(t_0) = x_0$$

where x is an n vector, f is a C^2 vector valued function and u is a r vector valued measurable control, is said to be <u>locally controllable</u> along a solution φ^v corresponding to control v if for some $t_1 > t_0$ all points in some state space (n dimensional) neighborhood of $\varphi^{v}(t_1)$ are attainable in time t_1 by trajectories of (2-9) with admissible control.

It would be somewhat falacious to say that a time dependent system is locally controllable, say at the origin, if all points in a neighborhood of the origin in state space are attainable by trajectories of the system in finite time. To see this, we consider the following example of G. Haynes.

Example 1:

$$\dot{x}_1 = -x_2 + (\cos t)\, u \quad , \qquad x(0) = 0, \ |u(t)| \leq 1$$
$$\dot{x}_2 = x_1 + (\sin t)\, u \ .$$

An integral of the motion is seen to be $x_1 \sin t - x_2 \cos t = 0$, which one can picture as a rotating (with time) line in x_1, x_2 space. As t varies from 0 to 2π, all points of E^2 are swept out by this line. Now multiply the first equation by cos t, the second by sin t and one obtains by adding;

$$\frac{d}{dt}(x_1 \cos t + x_2 \sin t) = u \quad \text{or}$$

$$x_1 \cos t + x_2 \sin t = \int_0^t u(\tau)\, d\tau \ .$$ Combining this with the integral of the motion gives

$$x_1^2(t) + x_2^2(t) = \left[\int_0^t u(\tau)\, d\tau\right]^2$$ implying that as time increases, the two dimensional neighborhoods of the origin of E^2 which are attainable also increase.

Since all solutions lie on a surface in (t, x) space, one would hardly feel that the system should be termed locally controllable and is <u>not</u> locally controllable by the definition given above.

We next proceed with an analysis, similar to that used in the papers [9] and [10], to examine local controllability about a given trajectory of the system (2-1). Let $x(t_o) = 0$ be initial data for this system v an arbitrary $\mathcal{L}_2$ control and φ^v the corresponding solution. Let $u(t; \xi)$, $\xi \in E^n$, be a family of controls such that $u(t; 0) = v(t)$, u_ξ exists, and denote $x(\cdot ; \xi)$ as the response to $u(\cdot , \xi)$. Then $x(\cdot ; \xi)$ satisfies

$$x(t; \xi) \equiv \int_{t_o}^{t} \left[g(\tau, x(\tau; \xi)) + H(\tau, x(\tau; \xi))\, u(\tau; \xi) \right] d\tau .$$

$$x_\xi (t; 0) \equiv \int_{t_o}^{t} \left[g_x(\tau, \varphi^v(\tau)) + H_x(\tau, \varphi^v(\tau)) v(\tau) \right] x_\xi (\tau, 0)$$

$$+ H(\tau, \varphi^v(\tau))\, u_\xi (\tau, 0) d\tau$$

where $H_x v$ is an nxn matrix with i jth element $\sum_{\nu = 1}^{r} H^i_{x_j}\, v^\nu$.

For each $t \geq t_o$, we view $x(t; \xi)$ as a mapping $\xi \longrightarrow x$ with $0 \longrightarrow \varphi^v(t)$. Let $Z(t; \varphi^v, u_\xi)$ denote the Jacobian matrix $x_\xi (t; 0)$. We have: <u>If for some $\bar{t}$, u_ξ, $Z(\bar{t}; \varphi^v, u_\xi)$ is non-singular, the attainable set at $\bar{t}$ contains a neighborhood of the point $\varphi^v(\bar{t})$</u>. Let $\Phi(t, t_o)$ be a fundamental solution matrix of the system

$$\dot{x}(t) = \left[g_x(t, \varphi^v(t)) + H_x(t, \varphi^v(t)) v(t) \right] x(t). \quad \text{Then}$$

$$Z(t; \varphi^v, u_\xi) \equiv \int_{t_o}^{t} \Phi(t,\tau)\, H(\tau, \varphi^v(\tau))\, u_\xi(\tau; 0)\, d\tau .$$

From lemma I.1 and corollary I.1 we have

Theorem II.2 (Kalman) A necessary and sufficient condition that there exist an rxn matrix u_ξ such that $Z(t_1; \varphi^v, u_\xi)$ is non-singular for some $t_1 > t_o$ is that the linear system

$$\dot{y}(t) = \left[g_x(t, \varphi^v(t)) + H_x(t, \varphi^v(t))v(t)\right] y(t) + H(t, \varphi^v(t))u(t)$$

is completely controllable.

In terms of the pfaffian approach the equivalent theorem is

Theorem II.3 A necessary and sufficient condition that there exist an rxn matrix u_ξ such that $Z(t_1, \varphi^v, u_\xi)$ is non-singular for some $t_1 > t_o$, is that the pfaffian system $B(t, \varphi^v(t))dx - B(t, \varphi^v(t))$ $\left[g_x(t, \varphi^v(t)) + H_x(t, \varphi^v(t))v(t)\right] x\, dt = 0$ be non-integrable, for some $t_1 \geq t_o$, i.e., that

$$b(t, \varphi^v(t))dx - b(t, \varphi^v(t)) \left[g_x(t, \varphi^v(t)) + H_x(t, \varphi^v(t))\, v(t))\right] x\, dt \tag{2-10}$$

is not and exact differential for any b which is a linear combination of the rows of B.

The same method, when applied to a system of the form (2-9) yields

Theorem II.3' A sufficient condition that there exists a $t_1 \geq t_o$ such that all points in some state space neighborhood of $\varphi^v(t_2)$ for all $t_2 > t_1$ are attainable in time t_2 by trajectories of (2-9) with admissible controls, is that there exists a $t_1 \geq t_o$ such that the pfaffian system

$$B(t; v)dy - B(t; v)\, f_x(t, \varphi^v(t), v(t))y\, dt = 0$$

is not integrable at t_1. [The notation $B(t; v)$ is used to denote the dependence of B on the reference trajectory, specifically $B(t; v)\, f_u(t, \varphi^v(t), v(t)) \equiv 0$.]

It is interesting at this point to see the implications of the assumption that (2-10) is an exact differential. This implies and is implied by

$$(2\text{-}11) \qquad \frac{d}{dt} b(t, \varphi^v(t)) \equiv -b(t, \varphi^v(t)) \left[g_x(t, \varphi^v(t) + H_x(t, \varphi^v(t))v(t) \right],$$

which can be recognized as the so-called adjoint system of the maximum principle [11] approach to the time optimal problem for system (2-1). It should be noted that if $b(t, \varphi^v(t))$ satisfies (2-11), then it is an adjoint vector which is orthogonal to all of the columns of H. Since the maximum principle (for control components bounded by one in absolute value) implies: choose $u^j(t) = \operatorname{sgn} \sum_{i=1}^{n} b^i(t, \varphi^v(t))H^{ij}(t, \varphi^v(t))$; in this case it yields no information.

I shall designate such a problem as one which admits a totally singular arc φ^v, i.e., where the maximum principle yields no information in the time optimal problem, for any components of the optimal control. The arc would be singular, but not totally singular, if there is an adjoint vector orthogonal to some, but not all columns of H.

<u>Theorem II.4</u> The pfaffian form (2-10) is an exact differential if and only if φ^v is a totally singular arc.

<u>Proof</u>: It has been shown above that if (2-10) is an exact differential, then the vector b satisfies (2-11), which implies φ^v is a totally singular arc.

If φ^v is a totally singular arc, there exists a vector p(t) such that i) $p(t)\, H(t, \varphi^v(t)) \equiv 0$ and ii) $\dot{p}(t) = -p(t) \left[g_x(t, \varphi^v(t)) + H_x(t, \varphi^v(t))\, v(t)\right]$. From i) we conclude that p(t) is a linear combination of the rows of $B(t, \varphi^v(t))$, while II) implies that this linear combination, (2-10), is an exact differential. ∎

To summarize; φ^v <u>not</u> a totally singular arc implies the pfaffian form (2-10) is <u>not</u> an exact differential which implies there exist $\bar{t} \geq t_o$ and $u_{\underline{\xi}}$ such that $Z(\bar{t}, \varphi^v, u_{\xi})$ is non-singular and the attainable set at time $\bar{t}$ contains a neighborhood of the point $\varphi^v(\bar{t})$. The contrapositive of this statement provides an interesting characterization of totally singular arcs, i.e., if for every $t_1 > t_o$ there exist points in every state space neighborhood of $\varphi^v(t_1)$ which are not attainable in time t_1 with $\mathscr{L}_2$ controls, the arc φ^v is totally singular. On the other hand, as will be shown by example, a totally singular arc can remain on the boundary of the attainable set, and thus provide a time optimal trajectory.

<u>Theorem II.5</u> If the system (2-1) is not completely controllable at t_o, $Z(t, \varphi^v, u_{\xi})$ is singular for all $t \geq t_o$, u_{ξ} and all reference trajectories φ^v, i.e., every trajectory φ^v is totally singular.

Proof: Any vector b, which is a linear combination of the rows of B, satisfies $b(t, x)H(t, x) \equiv 0$. Thus for any vector $v(t)$, $\frac{\partial}{\partial x}\left[b(t, x)H(t, x)v(t)\right] \equiv 0$, or $v(t)\, H^T(t, x)b_x(t, x) \equiv -b(t, x)H_x(t, x)v(t)$. Evaluating this identity at the point $(t, \varphi^v(t))$, substituting into (2-11) and expanding of the left side yields

$$(2\text{-}12) \quad b_t(t, \varphi^v(t)) + b(t, \varphi^v(t))g_x(t, \varphi^v(t)) + g(t, \varphi^v(t))b_x^{\,T}(t, \varphi^v(t)) \equiv v(t)H^T(t, \varphi^v(t))\left[b_x(t, \varphi^v(t) - b_x^{\,T}(t, \varphi^v(t))\right] .$$

This identity provides a necessary and sufficient condition that (2-10) be an exact differential, i.e., that φ^v be totally singular.

Now assume the system (2-1) is not completely controllable. This means that for some b, a linear combination of the rows of B, the pfafian form $b(t, x)dx - b(t, x)g(t, x)dt$ is an exact differential, or

$$b_t(t, x) \equiv -b(t, x)\, g_x(t, x) - g(t, x)\, b_x^{\,T}(t, x)$$
$$b_x(t, x) \equiv -\, b_x^{\,T}(t, x) \equiv 0.$$

Evaluating these two identities at $(t, \varphi^v(t))$ for an arbitrary control v shows that (2-12) is satisfied, hence every trajectory φ^v is totally singular.

A conjecture which one might be tempted to make is that if the system (2-1) is completely controllable, it admits no totally singular arcs. This is <u>not</u> true, as the following example from [2] shows.

Example II.1

$$\dot{x}_1 = x_1^2 - x_1^2 x_2 u \qquad\qquad x_1(0) = 1$$

$$\dot{x}_2 = -x_2 + u \qquad\qquad x_2(0) = 0 .$$

For the time optimal problem of reaching the point (2, 0), it is shown in [2] that $u \equiv 0$ is the optimal control, if the restriction $|u(t)| \leq 1$ is imposed, and it easily follows that this is also optimal in the class of $\mathcal{L}_2$ controls.

For this problem, one can use for the matrix B, the single vector $b = (1, x_1^2 x_2)$. The associated pfafian equation is

$$dx_1 + x_1^2 x_2\, dx_2 + x_1^2 (x_2^2 - 1)dt = 0.$$

Let $x = (x_1, x_2)$, $a(x) = (1, x_1^2 x_2, x_1^2(x_2^2 - 1))$. Then $(\text{curl } a(x)).$ $a(x) = 2\, x_2\, x_1^2 \neq 0$, thus the pfaffian is not integrable.

The optimal path from the point (1, 0) to $(\alpha, 0)$, $\alpha > 1$, is obtained with control $u \equiv 0$, and is

$$\varphi^o(t) \begin{cases} \frac{1}{1-t} \\ 0 . \end{cases}$$ This is a totally singular arc. To show this,

we note $b(t, \varphi^o(t)) \equiv (1, 0)$.

$$b(t, \varphi^o(t))dx - b(t, \varphi^o(t)) \left[g_x(t, \varphi^o(t)) + H_x(t, \varphi^o(t)) \circ 0 \right] x\, dt$$

$$= dx_1 + 0\, dx_2 - \frac{2x_1}{1-t}\, dt.$$

Let $\overline{a}(x, t) \equiv (1, 0, \frac{-2\, x_1}{1-t})$. Then $(\text{curl}\ \overline{a}) \cdot \overline{a} \equiv 0$ which implies the pfaffian $dx_1 + 0\, dx_2 - \frac{2\, x_1}{1-t}\, dt = 0$ is integrable, and φ^0 is a totally singular arc. Here the arc φ^0 is on the boundary of the attainable set.

It should be stressed at this point that it has not been shown that if for some control v, the matrix $Z(t, \varphi^v, u_\xi)$ is singular for all $t \geq t_0$, and u_ξ then sufficiently small n neighborhoods of a point $\varphi^v(t)$ contain points not attainable in time t, from initial data 0 given at t_0. In fact it will next be shown (Example II.2) that this is not the case. To do this we must produce a time optimal problem which posseses a totally singular arc which yields neither a maximum or minimum. Since the arc is totally singular, Theorem II.4 shows that one cannot conclude that the system is locally controllable along this arc by considering the linearized equations as in Theorem II.2. However the use of theorem II.3' on certain arcs which differ from the singular arc but have some points in common with it, will establish the local controllability.

We consider control systems of the form studied in [2], i.e.,

$$\text{(2-13)} \qquad \begin{aligned} \dot{x}_1(t) &= A_1(x(t)) + B_1(x(t))\, u(t) & \qquad x(0) &= x_0 \\ \dot{x}_2(t) &= A_2(x(t)) + B_2(x(t))\, u(t) & \qquad |u(t)| &\leq 1. \end{aligned}$$

We assume that in some region of interest $\mathcal{D}$ of state space,

$$\text{(2-14)} \qquad \Delta(x) \cong -B_2(x)\, A_1(x) + B_1(x)\, A_2(x) \neq 0$$

and that A_i, B_i, $i = 1, 2$ are C^1 in $\mathcal{D}$.

The pfaffian system associated with (2-13) is the single pfaffian equation

(2-15) $$B_2(x)\,dx_1 - B_1(x)\,dx_2 + \Delta(x)\,dt = 0.$$

Since $\Delta(x) \neq 0$ and multiplication by a factor does not change integrability, this can be rewritten as

(2-16) $$\frac{B_2(x)}{\Delta(x)}\,dx_1 - \frac{B_2(x)}{\Delta(x)}\,dx_2 + dt = 0.$$

Let $Z(x) = \left(\frac{B_2(x)}{\Delta(x)}, -\frac{B_1(x)}{\Delta(x)}, 1\right)$; then a necessary and sufficient condition that the pfaffian (2-16) be integrable at a point (t, x) is that $Z(x) \cdot \text{curl } Z(x) \equiv 0$ in a neighborhood of x. Computing yields

$$Z(x) \cdot \text{curl } Z(x) \equiv -\left[\frac{\partial}{\partial x_1}\left(\frac{B_1(x)}{\Delta(x)}\right) + \frac{\partial}{\partial x_2}\left(\frac{B_2(x)}{\Delta(x)}\right)\right] \equiv -\omega(x),$$

where $\omega(x)$ (using the notation of [2]) can be directly computed from the right sides of the differential equations (2-13).

Let v be a continuous control (this is sufficient continuity when the control appears linearly) satisfying $|v(t)| < 1$, and let φ^v be the corresponding trajectory of (2-13).

<u>Theorem II.6</u> If for some $t_1 \geq t_0$, $\varphi^v(t_1)$ is not a zero of ω, then for any $t_2 > t_1$ all points in some state space neighborhood of $\varphi^v(t_2)$ are attainable by trajectories of (2-13), in time t_2, with admissible controls.

Proof: The variational equation for the system (2-13) about the trajectory φ^v is given by

$$\dot{y}(t) = \left[A_x(\varphi^v(t)) + v(t)\, B_x(\varphi^v(t))\right] y(t) + B(\varphi^v(t))\, u(t)$$

where $A = \begin{pmatrix} A_1 \\ A_2 \end{pmatrix}$, $B = \begin{pmatrix} B_1 \\ B_2 \end{pmatrix}$. The pfaffian equivalent to (2-10) for this variational equation is

$$\text{(2-17)} \quad B_2(\varphi^v(t))dy_1 - B_1(\varphi^v(t))dy_2 + (-B_2(\varphi^v(t)), B_1(\varphi^v(t)))\left[A_x(\varphi^v(t)) + v(t)\, B_x(\varphi^v(t))\right] y\, dt = 0.$$

A sufficient condition that (2-17) be not integrable at t_1 is that

$$\text{(2-18)} \quad \frac{d}{dt}(B_2(\varphi^v(t)), B_1(\varphi^v(t)))\Big|_{t=t_1} \neq (-B_2(\varphi^v(t_1)), B_1(\varphi^v(t_1)))\left[A_x(\varphi^v(t_1)) + v(t_1)\, B_x(\varphi^v(t_1))\right],$$

which is implied by $\omega(\varphi^v(t)) \neq 0$ as can be shown by a straightforward calculation. [In terms of Theorem II.4, (2-18) states that $\varphi^v(t_1)$ is not a point of a singular arc. In [?, pg. 97] it is shown that for systems of this type singular arcs are characterized by the fact that ω is zero along them. It follows that if $\varphi^v(t_1)$ is not a zero of ω, then it is not a point of a singular arc, hence (2-17) is not integrable and the conclusion of the theorem follows.] ∎

It should be stressed that the integrability of (2-16) requires $\omega(x) = Z(x) \cdot \text{curl } Z(x)$ to be zero in a neighborhood of a point, while Theorem II.6 deals only with the value of ω at a point. It is possible, Example II.1, to have the pfaffian (2-16) not integrable at a point $(\bar{t}, \bar{x})$ at which $\omega(\bar{x}) = 0$, and yet have a trajectory φ^v such that $\varphi^v(\bar{t}) = \bar{x}$ and the system is not locally controllable about φ^v.

We next give the example of a problem which is locally controllable along a totally singular arc.

Example II.2 (A singular arc $\varphi^o(t)$ such that all points in a neighborhood of $\varphi^o(t_1)$ are attainable in time t_1.)

Consider the system

$$\dot{x}_1 = u \qquad |u(t)| \leq 1$$

$$\dot{x}_2 = 1 + x_2 x_1^2 u \qquad x(0) = 0$$

Then $\Delta(x) = 1$, $\omega(x) = x_1^2$, hence if we were to consider the time optimal problem of reaching the final point $x_f(0, \frac{1}{2})$, the Greens theorem approach [2], yields the following

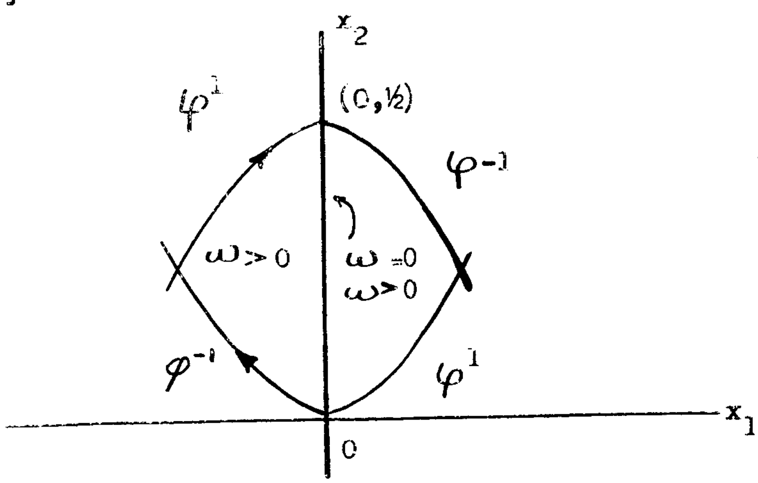

Figure 1

the optimal arc being shown by the arrows. There is an arc along which $\omega = 0$, i.e., $x_1 \equiv 0$, and while this can be attained with the control $u \equiv 0$ it yields neither a maximum or minimum to the time optimal problem. This arc we designate as φ^o;

$$\varphi^o(t) = \begin{cases} \varphi_1^o\ (t) \equiv 0 \\ \varphi_2^o\ (t) \equiv t. \end{cases}$$

It is easily checked that the variational equation along φ^o is <u>not</u> completely controllable.

Now consider a relation $x_1 = k_1 \sin k_2 x_2$, k_1, $k_2 > 0$ with $k_2 > 4\pi$. It will be shown that for k_1 sufficiently small, there exists a unique admissible continuous control $\bar{u}(t)$ with trajectory $\varphi^{\bar{u}}$ which has $\left\{(x_1, x_2):\ x_1 = k_1 \sin k_2 x_2,\ x_2 \geq 0\right\}$ as its track.

From the Greens theorem approach [2] and the symmetry of $\omega(x)$ about the line $x_1 = 0$, the parametrization of $\varphi^{\bar{u}}$ must be such that the even numbered crossings of the x_2 axis, counting only crossings which occur for $x_2 > 0$, one must have

$$\varphi_1^{\bar{u}}\left(\frac{2n\pi}{k_2}\right) = 0 = \varphi_1^o\left(\frac{2n\pi}{k_2}\right)$$

$$\varphi_2^{\bar{u}}\left(\frac{2n\pi}{k_2}\right) = \frac{2n\pi}{k_2} = \varphi_2^o\left(\frac{2n\pi}{k_2}\right).$$

We will be interested in the case $n = 1$, so that $\frac{2\pi}{k_2} < 1/2$. It will be shown that there is local controllability along $\varphi^{\bar{u}}$, and since

$\varphi^{\bar{u}}\left(\frac{2\pi}{k_2}\right) = \varphi^{o}\left(\frac{2\pi}{k_2}\right)$, it will follow that a neighborhood of $\varphi^{o}\left(\frac{2\pi}{k_2}\right)$ is attainable in time $\frac{2\pi}{k_2}$.

First we will show that for k_1 sufficiently small, there is a unique continuous u which leads to a trajectory $\varphi^{\bar{u}}$ having $\left\{(x_1, x_2):\ x_1 = k_1 \sin k_2 x_2,\ x_2 \geq 0\right\}$ as its track. Differentiation of the track relation with respect to t yields

$$\dot{x}_1(t) = k_1 k_2 \left[\cos k_2 x_2(t)\right] \dot{x}_2(t).$$

Substitution from the system equations leaves

$$u(t) = k_1 k_2 \left[\cos k_2 x_2(t)\right] \left[1 + x_2(t) x_1^{2}(t) u(t)\right]. \tag{2-19}$$

For any control u,

$$x_1(t) = \int_0^t u(\tau)\, d\tau$$

$$x_2(t) = \exp\left[\int_0^t u(\tau)\left(\int_0^\tau u(\sigma)\, d\sigma\right)^2 d\tau\right] \int_0^t \exp\left\{-\int_0^\tau u(\sigma)\left[\int_0^\sigma u(\gamma)\, d\gamma\right]^2 d\gamma\right\} d\tau .$$

Substituting these in (2-19) yields an expression of the form

$$u(t) = k_1 (\mathcal{F} u)(t)$$

where the definition of the nonlinear operator $\mathcal{F}$ is obvious. Let

$C\left[0, \tfrac{1}{2}\right]$ denote the space of continuous vector valued functions u on the interval $\left[0, \tfrac{1}{2}\right]$, with the supremum norm, and $B^{1/2}$ the closed ball of radius ½ in this space. It is easily shown that for k_1 sufficiently small but positive, $u \in B^{1/2} \Longrightarrow k_1 \mathcal{F} u \in B^{1/2}$, and $k_1 \mathcal{F}$ is a contracting map. Thus $k_1 \mathcal{F}$ has a unique fixed point in $B^{1/2}$, call this point $\bar{u}$. Then $\varphi^{\bar{u}}$ is <u>not</u> a singular trajectory, since k_1 positive implies $\bar{u}(t) \not\equiv 0$, and $\varphi^{\bar{u}}$ has the desired track.

Now for $0 < t_1 < \frac{\pi}{k_2}$, $\varphi^{\bar{u}}(t_1)$ is not a point of the singular arc, hence not a zero of ω. From Theorem II.6 it follows that all points in some neighborhood of $\varphi^{\bar{u}}(t_2)$, for any $t_2 > t_1$ are attainable in time t_2 by trajectories with admissible controls, hence this is true for $t_2 = \frac{2\pi}{k_2}$.

To determine local controllability along $\varphi^{\bar{u}}$ by use of the fundamental solution of the variational equation about this trajectory would be a virtually impossible task.

In concluding, it should be noted that totally singular arcs were defined with no mention made of transversality conditions. It is possible to use these conditions, in very special cases, to rule out the existence of singular arcs in the optimal strategy. Also, for a time optimal problem for a system of the form

$$\text{(2-20)} \qquad \dot{x}(t) = g(x(t)) + H(x(t))u(t)$$

the maximum principle yields the fact that the Hamiltonian is constant along the optimal path. We shall show that this cannot be used to rule out totally singular arcs, since such arcs automatically satisfy the condition even though the Hamiltonian is seemingly a function of time along them.

For the system (2-20) with any given control u(t) we define the Hamiltonian for the time optimal problem as

$$\mathcal{H}(t, x, p) \equiv p \cdot g(x) + p \cdot H(x)\, u(t) + 1.$$

A necessary condition is that $\mathcal{H}$ is a constant along the optimal trajectory, it need not be so on a non-optimal trajectory. Define the adjoint system as

$$(2\text{-}21) \qquad \dot{p}(t) = -p(t)\, g_x(x,(t)) - p(t)\, H_x(x(t))u(t)$$

<u>Theorem II.7</u> The Hamiltonian for the system (2-20) is constant along any totally singular arc.

<u>Proof:</u> We defined a totally singular arc as an arc φ^u which satisfies (2-20) for which there exists and adjoint vector p(t) satisfying (2-21) such that $p(t)H(\varphi^u(t)) \equiv 0$ for a set of t values having positive measure. Then

$$(2\text{-}22) \quad \frac{d}{dt}\mathcal{H}(t, \varphi^u(t), p(t)) \equiv \frac{d}{dt}\left[p(t) \cdot g(\varphi^u(t)) + 1\right] \equiv \dot{p}_i g^i + p_i g^i_{x_\nu} \dot{\varphi}_\nu .$$

From (2-20) $g^i \equiv \dot{\varphi}_i^u - H^{ik} u_k$.

From (2-21) $p^i g^i_{x_\nu} = - \dot{p}_\nu - p_i H^{ik}_{x_\nu} u_k$. Substituting in (2-22)

$$\frac{d}{dt}\mathcal{H}(t,\varphi^u(t),p(t)) \equiv \dot{p}_i\left[\dot{\varphi}_i^u - H^{ik}u_k\right] + \left[-\dot{p}_\nu - p_i H^{ik}_x u_k\right]\dot{\varphi}_\nu^u$$

$$= \left[-\dot{p}_i H^{ik} - p_i H^{ik}_{x_\nu}\dot{\varphi}_\nu^u\right] u_k \equiv -\left\{\frac{d}{dt}\left[p(t)H(\varphi^u(t))\right]\right\} u = 0$$

from the condition $p(t)H(\varphi^u(t)) = 0$. ∎

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III. THE EQUIVALENCE AND APPROXIMATION OF CONTROL PROBLEMS

INTRODUCTION TO SECTION III

In this section we will be concerned with the time optimal feedback control problem for an n vector system of the form

$$(3\text{-}1) \qquad \overset{\circ}{x}(t) = f(t, x(t), u(t)) \quad , \qquad \left(\overset{\circ}{x} = \frac{dx(t)}{dt}\right)$$

where the control u is an r vector valued function with values in a given set U. The major interest will be in feedback controls.

One of the difficulties in the theory of optimal feedback control is the discontinuity of the control with respect to the state variables, which the necessary condition termed the maximum principle, so often shows to be the case. Letting $H(t, x, p, u) \equiv p \circ f(t, x, u) - 1$; $u^*(t, x, p)$ be so that $H(t, x, p, u^*(t, x, p)) \geq H(t, x, p, u)$ for all $u \in U$, and $H^*(t, x, p) \equiv H(t, x, p, u^*(t, x, p))$, the Hamilton-Jacobi equation approach [1] often leads to a partial differential equation with discontinuous coefficients, while the Hamiltonian equations of motion which describe the system (the characteristic equations of the Hamilton-Jacobi equation) are of the form

$$(3\text{-}2) \qquad \overset{\circ}{x} = \frac{\partial}{\partial p} H^*(t, x, p) \quad , \qquad \overset{\circ}{p} = -\frac{\partial}{\partial x} H^*(t, x, p).$$

The maximum principle of Pontriagin, for time optimal problems, assures us that if $u^*(t)$ is an optimal control, $x^*(t)$ the corresponding optimal trajectory, then there exists an absolutely continuous <u>n vector</u> $p^*(t)$, not identically zero, such that $H^*(t,x^*(t),p^*(t)) \equiv H(t,x^*(t),p^*(t),u^*(t))$ while x^* and p^* satisfy equations (3-2). The usual use of the maximum principle proceeds, however, by attempting to generate candidates for an optimal trajectory by solving a two point boundary value problem for the system (3-2). Since u^* may be discontinuous, the fundamental questions of existence and uniqueness of solutions to these equations cannot easily be answered.

An alternative would be to restrict the controls to be continuous, or even C^1, (continuously differentiable) functions and attempt to generate within this class a sequence of controls which will in some sense tend toward the optimal control. In doing this, however, one must seemingly discard the maximum principle which is one of the most useful tools for generating optimal controls, for it so often demands discontinuous controls.

The approach taken here is not to forcefully restrict the class of approximating controls, but instead to generate a class of approximating problems whose solutions will be continuous or C^1 controls and will tend, in a given sense, to the solution of the original problem.

For the system (3-1) let $R(t, x) = \{f(t, x, u): u \in U\}$. We shall say that the time optimal problem for a system $\dot{x} = g(t, x, v)$, $v \in V$ is <u>equivalent</u> to that for the system (3-1) if $\{g(t,x,v): v \in V\} = R(t, x)$ for all (t,x) in some domain of interest. For given $\epsilon > 0$ we define the time optimal problem for the system $\dot{x} = h^{\epsilon}(t, x, v)$, $v \in V(\epsilon)$ to be an <u>ϵ-approximate equivalent</u> problem to the time optimal problem for (3-1) if $d\left(\{h^{\epsilon}(t,x,v): v \in V(\epsilon)\}, R(t, x)\right) < \epsilon$ for all (t,x) in the domain of interest. Here $d(Q,R)$ is the Hausdorff metric distance for sets in E^n.

Intuitively equivalent problems have the same optimal trajectories (as will be shown) while the optimal trajectories of ϵ- approximate equivalent problems will be close (uniformly) to those of the original problem.

It will be shown that under appropriate conditions (essentially the Fillipov existence conditions [2]) the approximating problems can be chosen in such a way that the corresponding feedback controls are continuous, or even of class C^1. In certain cases this allows the Hamilton-Jacobi theory, as derived in [1] , to be utilized for the construction of fields of optimal trajectories and optimal feedback controls.

Although we deal only with the time optimal problem, it should be noted that for a problem of the form $x'(\tau) = f(\tau, x(\tau), u(\tau))$, with the functional to be minimized being $\int_{\tau_o}^{\tau_f} \alpha(\sigma, x(\sigma), u(\sigma))d\sigma$ where the scalar valued function α satisfies $\alpha(\sigma, x, u) \geq \delta > 0$, the change of independent variable

$$t(\tau) = \int_{\tau_o}^{\tau} \alpha(\sigma, x(\sigma), u(\sigma))d\sigma$$ reduces the problem to an equivalent time optimal problem for the system

$$\dot{y}(t) = \left[\alpha(\tau(t), y(t), u(t))\right]^{-1} f(\tau(t), y(t), u(t)) \equiv g(t, y(t), u(t)).$$

THE MAXIMIZATION OF p.r WITH r IN A STRICTLY CONVEX SET

Our motivation is to choose approximating problems for which the maximum principle will yield smooth controls. Let $r^*(p)$ be the function which maximizes the functional $F(p,r) \equiv p \cdot r$ for fixed $p \in E^n - \{0\}$, $r \in R$ a given compact set in E^n. We begin by examining conditions on the set R which will insure that r^* is smooth since it is a maximization of this type which causes discontinuities in the control.

Definition. If S is a set contained in E^n (Euclidean n space) a support hyperplane is a hyperplane M which lies on one side of S and $S \cap M \neq \phi$, the empty set.

Definition. A convex set R contained in E^n will be said to be strictly convex if it contains more than one point, and every support hyperplane has at most one point in common with R.

If R is a compact set in E^n we denote its boundary by ∂R.

Lemma III.1. If R is a strictly convex set in E^n, then R has internal (interior) points. (This result depends on finite dimensionality).

Proof Let $r_o, r_1 \in R$, $r_o \neq r_1$, and V_1 be the linear variety of dimension one determined by these points. Let M_1 be any hyperplane containing V_1. Since M_1 contains two points of R it is not a support plane and there exists a point $r_2 \in R$, $r_2 \notin M_1$. Let V_2 be the linear variety determined by r_o, r_1 and r_2; V_2 has dimension two. Let M_2 be a hyperplane containing V_2. Again there is a point $r_3 \in R$, $r_3 \notin M_2$. We continue inductively getting at the (n-1)st step a linear variety V_{n-1} of dimension (n-1) determined by the points $r_o, \ldots, r_{n-1}$. Then there exists a unique hyperplane M_{n-1} containing V_{n-1}, and again a point $r_n \in R$, $r_n \notin M_{n-1}$. Since R is convex it contains the convex hull of the set of points $r_o, \ldots, r_n$; and since the vectors $r_1 - r_o$, $r_2 - r_o, \ldots, r_n - r_o$ are linearly independent, they determine an n cell which has non void interior.∎

Lemma III.2. Let R be a strictly convex, compact set in E^n. Then for any fixed $p \in E^n - \{0\}$, the function $F(p, \cdot)$ attains its maximum value at a unique point $r^*(p) = r_o \in \partial R$.

Proof For any fixed p, $F(p, \cdot)$ is a continuous function on the compact set R and hence attains its maximum there. Suppose the maximum is attained at an interior point $r_o \in R$. Let $N(r_o)$ be a neighborhood of r_o contained in R. Then $p \cdot r_o$ is an interior point of the real interval $p \cdot N(r_o) = \{p \cdot r: \ r \in N(r_o)\}$, contradicting the fact that F attains its maximum at r_o.

To show uniqueness, assume $F(p, \cdot)$ attains its maximum at r_o, while $r_1 \neq r_o$ belongs to R and $F(p, r_1) = F(p, r_o)$. Define $r(\alpha) = \alpha r_o + (1 - \alpha) r_1$, $-\infty < \alpha < \infty$. It follows that $F(p, r(\alpha)) = F(p, r_o)$ for every such point $r(\alpha)$. If for some α, $r(\alpha)$ is an interior point of R, the argument of the previous paragraph would show a contradiction to $F(p, \cdot)$ attaining its maximum at r_o. Thus the one dimensional linear variety $V = \{\alpha r_o + (1-\alpha)r_1: -\infty < \alpha < \infty\}$ does not intersect the interior of R, which is not empty by Lemma III.1. By theorem 3.6-E [3] there exists a closed hyperplane M containing V such that the interior of R lies strictly on one side of M. It follows that M is a support plane for R, and since M contains more than one point of R, this is a contradiction to the strict convexity. ▮

Theorem III.1 Let R be a strictly convex, compact set in E^n. Then the function $r^*(p)$ (shown to be well defined in lemma III-2) is continuous.

Proof Suppose $p_n \longrightarrow p$. Since R is compact, some subsequence of the sequence $\{r^*(p_n)\}$ converges to a point of R, and there is no loss of generality in assuming it is the original sequence, i.e.

let $r^*(p_n) \longrightarrow r_1$. We suppose $r^*(p) = r_2 \neq r_1$ and seek a contradiction. From the definition of r^*, $F(p, r_2) > F(p, r_1)$; let

$F(p, r_2) - F(p, r_1) = \delta > 0$.

Since F is continuous there exists an $N > 0$ such that

$|F(p_n, r_2) - F(p, r_2)| < \delta/4$ and $|F(p, r_1) - F(p_n, r^*(p_n)| < \delta/4$

for $n \geq N$. Then $F(p_n, r_2) - F(p_n, r^*(p_n)) \equiv \left[F(p, r_2) - F(p, r_1)\right]$ $+ \left[F(p_n, r_2) - F(p, r_2)\right] + \left[F(p, r_1) - F(p_n, r^*(p_n))\right] > \delta/2$ for $n \geq N$, a contradiction to the definition of $r^*(p_n)$. ∎

We next examine when the function $r^*(p)$ is C^1.

Definition. For $y \in E^n$, $|y| = \left\{\sum_1^n y_i^2\right\}^{1/2}$.

Lemma III.3. Let R be a strictly convex, compact set in E^n which has a unique outward unit normal $n(r)$ at each point $r \in \partial R$. Then for fixed $p \in E^n - \{0\}$, $F(p, \cdot)$ achieves its maximum at the unique point $r_o \in \partial R$ such that $n(r_o) = p/|p|$.

Proof Assume without loss of generality that zero is an interior point of R.

For $x \in E^n$, let $I(x) = \{a: a > 0, a^{-1}x \in R\}$ and define $\rho(x) = \inf_{a \in I(x)} a$; $\rho(x)$ is called the support function of R, or also the Minkowski functional. We note that if $r_o \in \partial R$ and y is any vector, then for a real scalar $\alpha > 0$, $\dfrac{\alpha y + r_o}{\rho(\alpha y + r_o)} \in \partial R$ and for α sufficiently small, is in a neighborhood of r_o.

From lemma III.2, we know $F(p, \cdot)$ achieves its maximum at a unique point on ∂R, let r_o be the point. Let $g(y, r_o) = \lim_{\alpha^+ \to 0} \frac{1}{\alpha}\left\{\frac{\alpha y + r_o}{\rho(\alpha y + r_o)} - r_o\right\}$.

Since ∂R has a unique outward normal at each point, $g(y, r_o) = -g(-y, r_o)$. Since $p \cdot r_o \geq p\ r$ for all $r \in \partial R$ in a neighborhood of r_o, it follows that $p \cdot g(y, r_o) \leq 0$ for all y. Assuming there exists y such that $p \cdot g(y, r_o) < 0$ implies $p \cdot g(-y, r_o) > 0$, a contradiction. Thus $p \cdot g(y, r_o) = 0$ for all y, or a necessary condition that r_o presents $F(p, \cdot)$ a maximum is that p be orthogonal to the support hyperplane at r_o. Since R is strictly convex it is easily shown that there are exactly two points which satisfy this necessary condition, one with outward normal $p/|p|$ giving F a maximum, the other with normal $-p/|p|$ which gives F a minimum. ∎

Definition. We say that a strictly convex, compact set R in E^n has a smooth boundary if there exists a unique outward unit normal $n(r) \in C^1$ defined on ∂R. (Actually we consider n as a restriction of a C^1 function in a neighborhood of $r \in \partial R$, see, for example, [4] pg. 27).

Theorem III.2. If R is a compact set in E^n with smooth boundary having positive Gaussian curvature at all points, then $r^*(p) \in C^1$.

Proof Since it is assumed that the unit normal to ∂R is of class C^1, the Gaussian curvature is a continuous positive function on ∂R. But ∂R is compact, thus the Gaussian curvature is bounded away from zero. From theorem 5.5 [5, pg. 35] it is easily followed that R is strictly convex.

From lemma III.3, we have thar $r^*(p)$ satisfies $n(r^*(p)) = p/|p|$. Let $r_o = r^*(p_o)$ be an arbitrary point on ∂R.

The method will be to utilize the implicit function theorem on a relation of the form $g(r, p) \equiv n(r) - p/|p|$.

Let $\zeta^1, \ldots, \zeta^{n-1}$ be a local coordinate system for a neighborhood of r_o on ∂R. Then the inclusion map from $\partial R \longrightarrow E^n$ determines n smooth functions $x_1(\zeta^1, \ldots, \zeta^{n-1}), \ldots, x_n(\zeta^1, \ldots, \zeta^{n-1})$ or briefly $x(\zeta)$. Assume $x(0) = r_o$ and let V_1 be a measurable neighborhood of zero in the local coordinate system.

Let S^{n-1} be the unit (n-1) sphere; we consider $n(\cdot): \partial R \longrightarrow S^{n-1}$. Define $\theta(\cdot): V_1 \longrightarrow S^{n-1}$ by $n(x(\zeta)) = \theta(\zeta)$. Thus
$n \in C^1 \Longrightarrow \theta \in C^1$.

Let $\varphi = \varphi(p) = p/|p|$, $p \in E^n - \{0\}$; then $\varphi \in C^1$. Our approach will be to utilize the implicit function theorem on the relation
$G(\zeta, \varphi) = \theta(\zeta) - \varphi$.

We note that $G \in C^1$, and if $\varphi_o = \varphi(p_o)$ then $G(0, \varphi_o) = 0$. Also $G_\zeta(0, \varphi_o) = \theta_\zeta(0)$. It must be shown that $\det(\theta_\zeta(0)) \neq 0$.

From differential geometry we recall that as ζ varies in V_1, $x(\zeta)$ traces out a region V_2 on ∂R while the normal $\theta(\zeta)$ traces out a region V_3 on the surface of the unit sphere. Let $K(\zeta)$ denote the Gaussian curvature of ∂R at $x(\zeta)$, and A_3 the "area" of V_3. Then

$A_3 = \int_{V_1} K(\zeta)d\zeta$. But $\int_{V_1} \det\left(\frac{\partial\theta(\zeta)}{\partial\zeta}\right)d\zeta = A_3$. Since V_1 is arbitrary (but measurable) and $\theta \in C^1$, this implies $\det\left(\frac{\partial\theta(\zeta)}{\partial\zeta}\right) = K(\zeta)$.

By assumption K is positive at all points of ∂R, hence $\det(\theta_\zeta(0)) \neq 0$. The implicit function theorem now gives the existence of a C^1 function $\zeta(\varphi)$ such that $G(\zeta(\varphi), \varphi) \equiv 0$.

Then $r^*(p) = x(\zeta(\varphi(p))) \in C^1$. ∎

The following is an example of a strictly convex set R with smooth boundary and a point at which the Gaussian curvature K is zero, for which $r^*(p)$ is not C^1.

Let part of the boundary of $R \subset E^2$ consist of the curve $y = x^4$, the rest so as to make R strictly convex and with smooth boundary. We restrict our attention to the defined part of the boundary, in particular to the point (0, 0) at which K is zero.

The outward normal is given by $(4x^3, -1)$. Let $p = (p_1, p_2)$ have p_2 negative and p_1 near zero. To compute $r^*(p) \equiv (x^*(p), y^*(p))$ we compute the point on the curve $y = x^4$ where the normal has direction numbers $(-p_1/p_2, -1)$. This gives $x^*(p) = (^-p_1/4\,p_2)^{1/3}$, $y^*(p) = (^-p_1/4p_2)^{4/3}$, and $\dfrac{\partial\, x^*(p)}{\partial\, p_1}$ is seen to not be continuous at $p_1 = 0$.

APPROXIMATION OF OPTIMAL TRAJECTORIES

The Time Optimal Problem

Consider the system (3-1), with U a compact set, and initial data $x(t_o) = x_o$. Let S be a smooth (C^2) manifold in the (n+1) dimensional (t,x) space with the property that for any t_2, t_3, $\{(t,x) \in S: t_2 \leq t \leq t_3\}$ is compact in E^{n+1}. The problem is to find a measurable function $u = u(t)$ having values in U, such that the solution of the initial value problem for (3-1) with $u = u(t)$, intersects the target S in minimum time; i.e., is an optimal trajectory.

We next give the conditions of Fillipov [2], which insure the existence of an optimal (open loop) control, and optimal trajectory for the time optimal problem.

Existence Conditions

(3-3) $f(t,x,u)$ is continuous in all variables t,x and u, and is continuously differentiable with respect to x.

(3-4) $x \cdot f(t,x,u) \leq C(|x|^2 + 1)$ for all t, x, u.

(3-5) $R(t,x) \equiv \{f(t,x,u): u \in U\}$ is convex for every t,x.

(3-6) There exists at least one measurable function $u(t)$ with values in U, such that the corresponding solution of the initial value, problem for (3-1) attains the target S for some $t_1 \geq t_o$.

Equivalence of Problems

Let the same time optimal problem, as posed for (3-1), also be posed for the system

(3-7) $\dot{x}(t) = g(t,x(t),v(t))$, $v(t) \in V$, a compact set,

where g satisfies condition (3-3). Let $Q(t,x) \equiv \{g(t,x,v): v \in V\}$.

Theorem III.3 Assume the existence conditions are satisfied for the time optimal problem for the system (3-1). Let $\varphi(\cdot\,; u^*)$ denote the optimal trajectory and u^* the optimal control. Then if $Q(t,x) = R(t,x)$ for all (t,x), $\varphi(\cdot\,; u^*)$ is an optimal trajectory for the time optimal problem for the system (3-7) and there exists a measurable function $v^*(t)$ with values in V such that $\overset{\circ}{\varphi}(t; u^*) = g(t, \varphi(t;u^*), v^*(t))$ almost everywhere.

Proof $f(t, \varphi(t; u^*), u^*(t))$ is a measurable function of t, with values (almost everywhere) in $R(t, \varphi(t; u^*))$, therefore in $Q(t, \varphi(t; u^*))$. From lemma 1 of Fillipov [2], there exists a measurable function $v^*(t)$ with values in V such that $f(t, \varphi(t;u^*), u^*(t)) = g(t, \varphi(t;u^*), v^*(t))$ almost everywhere. It follows that $\overset{\circ}{\varphi}(t; u^*) = g(t, \varphi(t;u^*), v^*(t))$ almost everywhere.

Now if $\varphi(\cdot\,; u^*)$ were not an optimal trajectory for (3-7), i.e., $\psi(\cdot\,; v)$ provides a better time, the same argument shows that $\psi(\cdot\,; v)$ is a solution of (3-1) for some measurable control u with values in U, thereby contradicting the assumed optimality of $\varphi(\cdot\,; u^*)$. ∎

This theorem stresses the fact that in seeking optimal trajectories, it is the set function $R(t,x)$ which is of major importance, not the function $f(t,x,u)$ or the control set U.

When the conditions of theorem III.3 are satisfied we define the time optimal problem for the system (3-7) to be equivalent to that for (3-1).

If the existence conditions are satisfied for the time optimal problem, from conditions (3-4) and (3-6) we can obtain a compact region of (t,x) space to which analysis can be restricted. Indeed for $t_o \leq t \leq t_1$ condition (3-4) implies any solution $x(t)$ of (3-1) satisfies $|x(t)|^2 \leq (|x_o|^2 + 1) \exp (2C\,|t_1 - t_o|)$. Here $|x(t)|$ stands for the usual Euclidean norm. Henceforth, we denote by $\mathcal{S}$ the compact region of (t,x)

space defined by $t_o \leq t \leq 2\, t_1$, $|x|^2 \leq (|x_o|^2 + 1)\exp(2C\,|2t_1 - t_o|)$.

Definition. The Hausdorff metric topology for non-empty compact sets in E^n is derived from the following metric: The distance between two non-empty compact sets X and Y in the smallest real number $d = d(X,Y)$ such that X lies in the d neighborhood of Y and Y lies in the d neighborhood of X.

ϵ Approximate Equivalent Problems

Definition. For given $\epsilon > 0$ the time optimal problem for the system $\dot{x} = h^{\epsilon}(t,x,v)$, h^{ϵ} continuous on $E^1 X E^n X V(\epsilon)$, is said to be an ϵ approximate equivalent problem to the time optimal problem for (3-1) if the set $R(t,x,\epsilon) \equiv \{h^{\epsilon}(t,x,v): v \in V(\epsilon)\} \supset R(t,x)$ and $d(R(t,x,\epsilon), R(t,x)) \leq \epsilon$ for all $(t,x) \in \mathcal{S}$.

Since $h^{\epsilon}(t,x,\cdot)$ is continuous on the compact set $V(\epsilon)$, $R(t,x,\epsilon)$ is compact.

Theorem III.4. Assume that the Fillipov conditions (3-3), (3-4) and (3-5) are satisfied for the time optimal problem with system equations (3-1). Then for every $\epsilon > 0$ there exists an ϵ approximate equivalent problem with system equations $\dot{x} = h^{\epsilon}(t,x,v)$, $v \in V(\epsilon)$ which satisfies the following properties.

a) The control set $V(\epsilon)$ can be taken to be the unit ball of E^n, which we denote B^n.

b) h^{ϵ} is a C^{∞} function on $\mathcal{S} \times B^n$, while for each $(t,x) \in \mathcal{S}$, $h^{\epsilon}(t,x,\cdot)$ is one-one on $B \longrightarrow E^n$.

c) The set $R(t,x,\epsilon) \equiv \{h^{\epsilon}(t,x,v): v \in B^n\}$ has smooth boundary having positive Gaussian curvature.

d) The (single valued) function $v^*(t,x,p)$ with values in B^n which maximizes $H(t,x,p,v;\epsilon) = p \cdot h^{\epsilon}(t,x,v) - 1$ for each $(t,x) \in \mathcal{D}$, $p \in E^n - \{0\}$, is C^1 in t, x, and p. Actually $v^*(t,x,p) \in \partial B^n = S^{n-1}$, the (n-1) sphere.

The proof will proceed by obtaining a simplicial approximation to $\mathcal{D}$ in which the diameters of the simplexes are sufficiently small. For each vertex (t_i, x_i) of a simplex, we approximate the convex set $R(t_i, x_i)$ by a strictly convex set $Q(t_i, x_i, \epsilon)$ having positive Gaussian curvature. A vector function $g^{\epsilon}(t_i, x_i; \cdot)$ is then constructed so that $Q(t_i, x_i; \epsilon) = \{g^{\epsilon}(t_i, x_i; v): v \in B^n\}$, and by use of g^{ϵ}, the set function Q is extended continuously to all of $\mathcal{D}$ in such a manner that for each $(t,x) \in \mathcal{D}$, $Q(t,x;\epsilon)$ has smooth boundary with positive Gaussian curvature. The desired function h^{ϵ} is then obtained by smoothing the function g^{ϵ} in the variables (t,x) via the Friedricks mollifier technique.

Proof $R(t,x)$ is continuous, in the Hausdorff metric topology, on the compact set $\mathcal{D}$. For any $\epsilon > 0$ let $\delta > 0$ be such that $d(R(t,x),R(t',x')) < \epsilon/8$ whenever $|(t,x)-(t',x')| < \delta$. Let σ_g^{n+1} be any bounded geometric simplex which contains $\mathcal{D}$, and K_g be the geometric complex consisting of this single simplex. By barycentric subdivision K_g can be subdivided into a geometric complex K_g' consisting of a family of geometric simplexes $\{\bar{\sigma}_g^{n+1}\}$, each having diameter less than δ.

Each point $(t,x) \in \mathcal{D}$ has a unique representation of the form

$$(t,x) = \sum_{i=1}^{n+2} \alpha_i (t_i, x_i) \quad \text{with } 0 \le \alpha_i \le 1, \quad \sum \alpha_i = 1; \text{ where the}$$

(n+2) points (t_i, x_i) are the vertices of the geometric simplex from the family $\{\bar{\sigma}_g^{n+1}\}$ to which the point (t,x) belongs. Without loss of generality we can now consider the union of the members of $\{\sigma_g^{n+1}\}$ which have all vertices in $\mathcal{A}$ as a new domain of interest; call this domain again $\mathcal{A}$.

Let (t_i, x_i) be an arbitrary vertex in $\mathcal{A}$. Then $R(t_i, x_i)$ is convex. Let $\mathcal{N}(R(t_i, x_i), \epsilon/4)$ be a convex $\epsilon/4$ neighborhood of $R(t_i, x_i)$. From [6, pg. 38] there exists a strictly convex set $Q(t_i, x_i, \epsilon)$ containing $\mathcal{N}(R(t_i, x_i), \epsilon/4)$; having an analytic boundary with positive Gaussian curvature, and such that $d(Q(t_i, x_i, \epsilon), \mathcal{N}(R(t_i, x_i), \epsilon/4)) < \epsilon/4$.
For each $(t_i, x_i) \in \mathcal{A}$ we construct a corresponding set $Q(t_i, x_i, \epsilon)$ as above. We next proceed to define a set valued function $Q(t, x, \epsilon)$ on all of $\mathcal{A}$.

It can be assumed without loss of generality that $0 \in R(t,x)$ for all $(t,x) \in \mathcal{A}$. Indeed if this were not so, one could choose a point $u_o \in U$ and construct new sets $S(t,x) \equiv \{ f(t,x,u) - f(t,x,u_o): u \in U \}$ which satisfy this property.

Let B^n be the unit ball in E^n; S^{n-1} its' surface and $v^1, \ldots, v^{n-1}$ a coordinate system on S^{n-1} while v^n measures distance from the origin. Then a ray from the origin through $(v^1, v^2, \ldots, v^{n-1}, 1)$ strikes $\partial Q(t_i, x_i, \epsilon)$ in a unique point which we denote $g^\epsilon(t_i, x_i, v^1, \ldots, v^{n-1}, 1)$. This defines $g^\epsilon(t_i, x_i, \cdot)$ on S^{n-1}; to extend it to B^n let $v = (v^1, \ldots, v^n)$ B^n. Define $g^\epsilon(t_i, x_i, v)$ as that point in $Q(t_i, x_i, \epsilon)$ which lies on the ray through the origin and $(v^1, \ldots, v^{n-1}, 1)$ and is such that

$$\frac{|g^\epsilon(t,x,v)|}{|g^\epsilon(t,x,v^1,\ldots,v^{n-1},1)|} = v^n.$$

Then $g^{\epsilon}(t_i, x_i, \cdot): B^n \longrightarrow Q(t_i, x_i, \epsilon)$ in a one to one fashion. We will define $Q(t, x, \epsilon)$ on all of $\mathcal{A}$ by extending the definition of g^{ϵ} to all $(t,x) \in \mathcal{A}$.

Assume $(t,x) \in \mathcal{A}$. Let $(t,x) = \sum_{i=1}^{n+2} \alpha_i (t_i, x_i)$ be the unique representation of (t,x) in terms of the vertices of the geometric simplex of K_g' to which it belongs. Define

$$g^{\epsilon}(t,x,v) = \sum_{i=1}^{n+2} \alpha_i\, g^{\epsilon}(t_i, x_i, v),\quad v \in B^n.$$ Then if

$Q(t,x,\epsilon) = \left\{ g^{\epsilon}(t,x,v) : v \in B^n \right\}$ it follows that:

i) $\eta(R(t,x), \epsilon/8) \subset Q(t,x,\epsilon)$. Indeed, from the choice of δ, $\eta(R(t,x), \epsilon/8) \subset \eta(R(t_i, x_i), \epsilon/4) \subset Q(t_i, x_i, \epsilon)$ for all vertices (t_i, x_i) of the simplex in which (t,x) is contained. But $Q(t,x,\epsilon) = \sum \alpha_i Q(t_i, x_i, \epsilon)$. Thus if a point is in $\eta(R(t, x), \epsilon/8)$ it is in $Q(t,x, \epsilon)$.

ii) $d(Q(t,x,\epsilon), R(t, x)) < 3\epsilon/4$. To show this one notes that $R(t_i, x_i) \subset \eta(R(t_j, x_j), \epsilon/4) \subset Q(t_j, x_j, \epsilon)$ for all $i, j = 1, 2, \ldots, n+2$. Therefore

$$\begin{aligned}
d(R(t,x),Q(t,x,\epsilon)) &\le d(R(t,x), R(t_i, x_i)) + \\
&\quad d(R(t_i, x_i), \sum_j \alpha_i Q(t_j, x_j, \epsilon)) \le \epsilon/8 + \\
&\quad \max_j \left[d(R(t_i, x_i), Q(t_j, x_j, \epsilon)) \right] \le \epsilon/8 + \\
&\quad \max_j \left[d(R(t_i, x_i), R(t_j, x_j)) + d(R(t_j, x_j), Q(t_j, x_j, \epsilon)) \right] \le 3\epsilon/4.
\end{aligned}$$

iii) $Q(t,x,\epsilon)$ is strictly convex, with smooth boundary having positive Gaussian curvature, for each (t,x) . Indeed of $K(t,x,v^1,\ldots,v^{n-1})$ is Gaussian curvature at the point $g(t,x,v^1,\ldots,v^{n-1}, 1) \in \partial R(t,x,\epsilon)$, then $K(t,x,v^1,\ldots,v^{n-1}) = \sum_{i=1}^{n+2} \alpha_i K(t_i, x_i, v^1,\ldots,v^{n-1})$.

iv) From the construction, $g^{\epsilon}(t,x,v)$ is analytic in v for fixed (t,x) and continuous in (t,x) for fixed v.

Combining the results of i) and ii) shows that for $(t,x) \in \mathcal{A}$,

$$\eta(R(t,x), \epsilon/8) \subset Q(t,x,\epsilon) \subset \eta(R(t,x), 3\epsilon/4).$$

It will next be shown that using $g^{\epsilon}(t,x,v)$ one can construct a mapping $h^{\epsilon}(t,x,v)$ on $\mathcal{A} \times B^n \longrightarrow E^n$ such that if $R(t,x,\epsilon) = \left\{ h^{\epsilon}(t,x,v) : v \in B^n \right\}$, then $R(t,x,\epsilon)$ is a strictly convex, compact set containing $R(t,x)$; $d(R(t,x,\epsilon), R(t,x)) < \epsilon$; $\partial R(t,x,\epsilon)$ is smooth with positive Gaussian curvature, and if $n(t,x,h^{\epsilon}(t,x,v^1,\ldots,v^{n-1},1)$ is a unit normal to $\partial R(t,x,\epsilon)$ at $h^{\epsilon}(t,x,v^1,\ldots,v^{n-1}, 1)$ then it is a C^1 function of all arguments.

For simplicity of notation let $y = (t,x)$ denote a point in $\mathcal{A}$, and let $S^k(y-\bar{y})$ be a mollifier function; see [7] . As an example one could choose

$$S^k(y-\bar{y}) = (k/4\pi)^{\frac{n+1}{2}} \exp\left\{ -\frac{k}{4} \left[\sum_{i=1}^{n+1} (y^i - \bar{y}^i)^2 \right] \right\}.$$

Extend $g^{\epsilon}(y,v)$ as the zero function for y in the complement of $\mathcal{A}$.

Define $$h^k(y,v) = \int_{E^{n+1}} S^k(y-\bar{y})\, g^{\epsilon}(\bar{y},v)\, d\bar{y} .$$

Then for every integer $k > 0$, h^k is an analytic function, while h^k and its derivatives with respect to v tend uniformly to g^{ϵ} and its derivatives with respect to v.

Let $R^k(t,x,\epsilon) \equiv \left\{h^k(t,x,v): v \in B^n\right\}$. Since the Gaussian curvature to $\partial Q(t,x,\epsilon)$ is given as a multilinear combination of the derivatives $g^{\epsilon}_{v^i}(t,x,v^1,\ldots,v^{n-1},1)$ while the curvature of $\partial R^k(t,x,\epsilon)$ is given by the same multilinear combination of the derivatives $h^k_{v^i}(t,x,v^1,\ldots,v^{n-1},1)$; one can choose k sufficiently large so that $\partial H^k(t,x,\epsilon)$ has positive Gaussian curvature while $R(t,x) \subset H^k(t,x,\epsilon) \subset \mathcal{N}(R(t,x),\epsilon)$. For such a choice of k, define $h^{\epsilon}(t,x,v) = h^k(t,x,v)$, $R(t,x,\epsilon) \equiv \left\{h^{\epsilon}(t,x,v): v \in B^n\right\}$.

From its construction, h^{ϵ} satisfies conclusions a), b) and c), while a unit normal $n(t,x,h^{\epsilon}(t,x,v^1,\ldots,v^{n-1},1))$ to $\partial R(t,x,\epsilon)$ is a C^1 function of $(t,x,v^1,\ldots,v^{n-1})$.

It remains to show part d). Using lemma III.3 define $r^*(t,x,p;\epsilon)$ as the unique point on $\partial R(t,x,\epsilon)$ such that $n(t,x,r^*(t,x,p,\epsilon)) = p/|p|$. It will be shown that r^* is a C^1 function of t, x, and p by a proof similar to that of theorem III.2. Defining $v^*(t,x,p)$ as the unique point on ∂B^n such that $h^{\epsilon}(t,x,v^*(t,x,p)) = r^*(t,x,p,\epsilon)$ it follows that v^* maximizes $H(t,x,p,v;\epsilon)$ and it will be shown that v^* is a C^1 in t, x and p.

For fixed (t,x), we have

$$S^{n-1} \xleftrightarrow{h^{\epsilon}(t,x,v^1,\ldots,v^{n-1},1)} \partial R(t,x,\epsilon) \xleftrightarrow{n(t,x,r)} S^{n-1}$$

which naturally induces a map $\Theta(t,x,v^1,\ldots,v^{n-1})$ from $S^{n-1} \longleftrightarrow S^{n-1}$ defined by $\Theta(t,x,v^1,\ldots,v^{n-1}) \equiv n(t,x,h^{\epsilon}(t,x,v^1,\ldots,v^{n-1},1))$. Since we are only interested in $\partial B^n = S^{n-1}$, no confusion should occur if for the remainder of this argument we let $v = (v^1,\ldots,v^{n-1}) \in S^{n-1}$ and therefore write $\Theta(t,x,v)$. This will be done.

Let $\varphi = \varphi(p) = p/|p|$, $p \in E^n - \{0\}$ and define $G(t,x,v,\varphi) \equiv \Theta(t, x, v,) - \varphi$. We will apply the implicit function theorem to G, which is easily seen to be a C^1 function. For each t_o, x_o, $\varphi_o = p_o/|p_o|$, there exists a unique point $r_o = r^*(t_o, x_o, p_o; \epsilon)$ such that if $n(t_o, x_o, r_o) = p_o/|p_o|$ and v_o is the unique point on S^{n-1} such that $h^\epsilon(t_o, x_o, v_o) = r_o$, then $G(t_o, x_o, v_o, \varphi_o) = 0$. One next notes that $G_v(t_o, x_o, v_o, \varphi_o) = \Theta_v(t_o, x_o, v_o)$, and from the definition of Θ (see also the proof of theorem III.2) $\det\left[\Theta_v(t_o, x_o, v_o)\right]$ is the Gaussian curvature at $r_o \in \partial R(t, x, \epsilon)$ which is positive. The implicit function theorem yields the existence of a C^1 function $v(t, x, \varphi)$ such that $G(t, x, v,(t, x, \varphi), \varphi) \equiv 0$ in a neighborhood of the arbitrary point t_o, x_o, φ_o. Then $r^*(t, x, p; \epsilon) \equiv h^\epsilon(t, x, v(t, x, \varphi(p))) \in C^1$, while $v^*(t, x, p) \equiv v(t, x, \varphi(p))$ is also C^1. ∎

THE RELATION OF TRAJECTORIES OF THE APPROXIMATING PROBLEM TO THOSE OF THE TIME OPTIMAL PROBLEM

We assume the system (3-1) satisfies the Fillipov existence conditions (3-3), (3-4), (3-5) and (3-6), with t_1 a time in which the target set S is attainable. For any $\epsilon > 0$ let $h^{\epsilon}(t, x, v)$, $v \in V(\epsilon)$, be an ϵ approximate equivalent problem (not necessarily having the special properties shown to exist in theorem III.4). From condition (3-6) and the relation $R(t, x, \epsilon) \supset R(t, x)$, it readily follows that for every $\epsilon > 0$ there exists at least one measurable function v with values in $V(\epsilon)$ such that the corresponding trajectory $\varphi^{\epsilon}(\cdot\, ; v)$ of the ϵ approximate problem attains the target S.

It will next be shown that when dealing with the approximate problem, analysis can again be restricted to a compact set. Indeed any vector $h^{\epsilon}(t, x, v)$ can be written as $f(t, x, u) + \alpha(t, x)$ where $|\alpha(t, x)| \leq \epsilon$. Then for any trajectory x(t) of the approximate problem

$$\frac{1}{2}\frac{d}{dt}|x(t)|^2 = x(t)\cdot h^{\epsilon}(t, x(t), v(t)) = x(t)\cdot f(t, x(t); u(t)) + x(t)\cdot \alpha(t, x(t))$$

$$\leq C(1 + |x(t)|^2) + \epsilon|x(t)|.$$

$$\frac{d}{dt} \ln(1 + |x(t)|^2) \leq 2C + \frac{2\epsilon|x(t)|}{1 + |x(t)|^2} \leq 2(C + \epsilon),$$

$$|x(t)|^2 \leq (1 + |x_o|^2)e^{2(C+\epsilon)(t_1 - t_o)}.$$

Define $\mathcal{A}^{\epsilon}$ to be the compact region in E^{n+1} dimensional (t,x) space so that $|x|^2 \leq (1 + |x_o|^2)\exp\left[2(C+\epsilon)(2t_1 - t_o)\right]$, $t_o \leq t \leq 2t_1$.

Theorem III.5. Consider a sequence $\{\epsilon_k\}$ with $\epsilon_k > 0$, $\epsilon_k \longrightarrow 0$ and let φ^{ϵ_k} denote the time optimal trajectory (assumed to exist) for the ϵ_k approximate problem. Then $\{\varphi^{\epsilon_k}\}$ is an equicontinous family on the interval $[t_o, t_1]$. It has a uniformly convergent subsequence which converges to a function φ^* having the following properties.

i) φ^* is absolutely continuous

ii) There exists a measurable function u^* with values in U such that $\dot{\varphi}^*(t) = f(t, \varphi^*(t), u^*(t))$ almost everywhere.

iii) There exists a smallest $t^* \geq t_o$ such that $\varphi^*(t^*) \in S$

iv) φ^* is a time optimal trajectory for the system (3-1).

<u>Proof</u> We shall prove the conclusions in the order that they are stated.

Without loss of generality, assume that $R(t,x,\epsilon_1) \supset R(t,x,\epsilon_2) \supset \ldots \supset R(t,x)$. Therefore analysis can be restricted to the compact region $\mathcal{A}^{\epsilon_1}$. Our first goal is to show that there is a constant N independent of ϵ_k such that φ^{ϵ_k} is Lipschitz continuous with Lipschitz constant N. To accomplish this, for a compact set R in E^n let $\rho(R)$ denote $\max_{r \in R} |r|$. For fixed ϵ_1, $R(t, x, \epsilon_1)$ is a continuous set valued function (in the Hausdorff metric topology) on the compact set $\mathcal{A}^{\epsilon_1}$ and therefore the composite map $\rho(R(t, x, \epsilon_1))$ is a continuous real valued function on $\mathcal{A}^{\epsilon_1}$, hence bounded. Let N be its bound. It follows that $|h^{\epsilon_k}(t, x, v)| \leq N$ for all ϵ_k and any trajectory φ^{ϵ_k} is Lipschitz continuous with Lipschitz constant N. Thus $\{\varphi^{\epsilon_k}\}$ is equicontinuous and has a subsequence which converges uniformly to a Lipschitz continuous function φ^*, which is therefore absolutely continuous. We will not distinguish between $\{\varphi^{\epsilon_k}\}$ and its convergent subsequence.

ii) We next show that for almost all $t \in [t_o, t_1]$, $\dot{\varphi}^*(t) \in R(t, \varphi^*(t))$.

Since the set function $R(t, x)$ is continuous in the Hausdorff metric topology (a consequence of the continuity of f), for any $\nu > 0$ let $R_\nu(t, x)$ be a closed convex ν - neighborhood of $R(t, x)$. Then $R_\nu(t, x)$ is also a continuous set function.

Since $\dot{\varphi}^{\epsilon_k}(t) \in R(t, \varphi^{\epsilon_k}(t), \epsilon_k)$ and $R(t, x, \epsilon_k) \longrightarrow R(t, x)$ in the Hausdorff metric topology, there exists and N such that for all $n \geq N$, $\dot{\varphi}^{\epsilon_k}(t) \in R_\nu(t, \varphi^*(t))$. Fillipov's proof of theorem 1, [2] now applies to show that for almost all t, $\dot{\varphi}^*(t) \in R_\nu(t, \varphi^*(t))$. But $R(t, x)$ is closed and ν arbitrarily small, hence $\dot{\varphi}^*(t) \in R(t, \varphi^*(t))$ for almost all t.

From the lemma of Fillipov [2], we then obtain the existence of a measurable control u* with values in U, such that for almost all $t \in [t_o, t_1]$, $\dot{\varphi}^*(t) = f(t, \varphi^*(t), u^*(t))$.

iii) Let $t_{\epsilon_k} > t_o$ denote the optimal time for the ϵ_k approximate problem. Since $R(t, x, \epsilon_1) \supset R(t, x, \epsilon_2) \supset \ldots$ it follows that $\{t_{\epsilon_k}\}$ is a monotone non-drecreasing sequence of reals bounded above by t_1. Let t* be its limit. Now $\varphi^{\epsilon_k}(t_{\epsilon_k}) \in S$ for each k, and $\{(t,x) \in S: t_o \leq t \leq t_1\}$ is compact in E^{n+1}, thus $\varphi^{\epsilon_k}(t_{\epsilon_k}) \longrightarrow \varphi^*(t^*) \in S$.

iv) Suppose φ^* is not a time optimal trajectory for the system (3-1). Then there exists a measurable control u with values in U and corresponding trajectory $\varphi(\cdot\,; u)$ such that $\varphi(t_o; u) = x_o$, $\varphi(t_3; u) \in S$ and $t_3 < t^*$. This implies that for k sufficiently large, $t_3 < t_{\epsilon_k}$; but $\varphi(\cdot\,; u)$ is an admissible trajectory to all ϵ approximate problems. This contradicts the optimality of φ^{ϵ_k}. ∎

This theorem essentially tells us that for sufficiently small ϵ, the optimal trajectories of the ϵ approximate problem are uniformly close to an optimal trajectory of the original problem.

In the next section the "smoothness" which theorem III.4 shows is possible for the feedback control of the ϵ approximate problem, will be exploited to obtain solutions.

Hamilton-Jacobi Theory

Let the time optimal problem for (3-1) satisfy the Fillipov existence conditions. Let $\dot{x} = h^{\epsilon}(t, x, v)$ denote an ϵ approximate system with the properties a), b), c) and d), shown to exist in theorem III.4. For the time optimal problem associated with the approximate problem we define the functions

$$H(t, x, p, v, \epsilon) \equiv p \cdot h^{\epsilon}(t, x, v) - 1$$

$$H^*(t, x, p, \epsilon) \equiv H(t, x, p, v^*(t, x, p), \epsilon).$$

The inequality

$$(3\text{-}8) \qquad H(t, x, p, v^*, \epsilon) > H(t, x, p, v, \epsilon) \text{ for all } v \in B^n, \ v \neq v^*$$

is a consequence of the definition of v^*.

For the sake of completeness we repeat a short argument of Kalman ([1], pp. 321-322) to show that for fixed $\epsilon > 0$,

$$H^*_x(t, x, p, \epsilon) \equiv p \cdot h^{\epsilon}_x(t, x, v^*(t, x, p))$$

$$H^*_p(t, x, p, \epsilon) \equiv h^{\epsilon}(t, x, v^*(t, x, p)).$$

Indeed, we know that $v^*(t, x, p) \in \partial B^n = S^{n-1}$, thus let $g(v)$ be a smooth relation such that $g(v) = 0$ determines S^{n-1} in a neighborhood of $v^*(t, x, p)$. Then $g_v(v^*(t, x, p))\, v^*_x(t, x, p) \equiv 0$ and $g_v(v^*(t, x, p))\, v^*_p(t, x, p) \equiv 0$.

Noting that v^* maximizes $H(t, x, p, v, \epsilon)$, we consider this maximization subject to the constraint $v \in S^{n-1}$, i.e., $g(v) \equiv 0$. The Lagrange multiplier rule implies $H_v + \nu g_v = 0$ where $\nu \neq 0$. Evaluting this at v^* and multiplying on the right by $v^*_x(t, x, p)$ and $v^*_p(t, x, p)$, in turn, gives the required result.

If φ^ϵ, ψ^ϵ are solutions, respectively, to the boundary value problem

$$(3\text{-}9) \qquad \dot{x} = H^*_p(t, x, p, \epsilon) \equiv h^\epsilon(t, x, v^*(t, x, p))$$

$$(3\text{-}10) \qquad \dot{p} = -H^*_x(t, x, p, \epsilon) \equiv -p \cdot h^\epsilon_x(t, x, v^*(t, x, p))$$

with boundary data $x(t_o) = x_o$, $x(t_1) = x_1$, then (3-8) shows that $v^*(t, \varphi^\epsilon(t), \psi^\epsilon(t))$ satisfies the necessary condition termed the maximum principle, for being an optimal (open loop) control for the time optimal problem of attaining the state x_1 from the state x_o for the approximating system.

It should be noted that under the conditions assumed, $v^* \in C^1$ and the initial value problem for the equations (3-9), (3-10) with data given at t_o will have a unique solution in a neighborhood of t_o. If v^* is discontinuous, this presents a serious difficulty in the application of the maximum principle.

With the (Hamiltonian) function $H^*(t, x, p, \epsilon)$, $\epsilon > 0$ and fixed, we associate the Hamilton-Jacobi partial differential equation

$$(3\text{-}11) \qquad V_t(t,x) + H^*(t, x, V_x(t, x), \epsilon) = 0.$$

Let the target S be a "smooth" n-dimensional, non-characteristic manifold in the (n+1) dimensional (t,x) space, and prescribe the Cauchy data $V(t,x) = 0$, $(t,x) \in S$. The solution, in the classical sense, of this partial differential equation problem, we denote by V^ϵ; the domain of solution by $\mathcal{D}(\epsilon, S)$.

The characteristic equations associated with (3-11) are the equations (3-9), (3-10). If a point (t_o, x_o) is in $\mathcal{S}(\epsilon, S)$ there exists a point $(t_1, x_1) \in S$ such that the boundary value problem consisting of the equations (3-9), (3-10) with boundary data for (3-10) being $x(t_o) = x_o$, $x(t_1) = x_1$, has a solution. The solution of such a boundary value problem, when it exists, will be denoted by φ^ϵ, ψ^ϵ.

From the continuity condition, for each $\epsilon > 0$, $V_x^\epsilon(t, \varphi^\epsilon(t))$ exists and satisfies equation (3-10). (See for example [1]). Thus we can make the association $\psi^\epsilon(t) \equiv V_x^\epsilon(t, \varphi^\epsilon(t))$.

Let $\mathcal{S}^-(\epsilon, S)$ denote the set of points $(t_o, x_o) \in \mathcal{S}(\epsilon, S)$ for which $t_o \leq t_1$; (t_1, x_1) being the point on S joined to (t_o, x_o) by a curve φ^ϵ. Assume $(t_o, x_o) \in \mathcal{S}^-(\epsilon, S)$. If we use the initial data $x(t_o) = x_o$, $p(t_o) = V_x^\epsilon(t_o, x_o)$; by virtue of knowing a solution of the partial differential equation we have the proper initial data to reduce the previous two point boundary value problem for (3-9) and (3-10) to an initial value problem. Thus to determine the trajectory φ^ϵ we can consider the system

$$(3\text{-}12) \qquad \dot{x} = H_p^*(t, x, V_x^\epsilon(t, x); \epsilon), \qquad x(t_o) = x_o.$$

The major advantage of this method is that now $v^* = v^*(t, x, V_x^\epsilon(t,x))$, i.e., a feedback control.

<u>Theorem III.6</u> (Kalman) Assume $(t_o, x_o) \in \mathcal{S}^-(\epsilon, S)$; V^ϵ is the solution of the Hamilton-Jacobi equation (3-11) and φ^ϵ the solution of (3-12). Then φ^ϵ is a time optimal trajectory relative to all trajectories $\varphi(\cdot\,; v)$ which connect (t_o, x_o) to S and lie in $\mathcal{S}^-(\epsilon, S)$.

Proof Assume, without loss of generality, that $(t_o, x_o) \notin S$.

From the definition of H, H* and V^ϵ,

$$0 \equiv V_t^\epsilon(t, x) + V_x^\epsilon(t, x) \cdot h^\epsilon(t, x, v^*(t, x, V_x^\epsilon(t, x))) - 1 > V_t^\epsilon(t, x) +$$

$$V_x^\epsilon(t, x) \cdot h^\epsilon(t, x, v) - 1 \qquad \text{for all } v \in B^n,\ v \neq v^*.$$

Assume that t_ϵ $(t_\epsilon > t_o)$ is the first time such that $(t_\epsilon, \varphi^\epsilon(t_\epsilon)) \in S$. Let Ω^ϵ denote the set of measurable control functions having values in B^n and leading to trajectories of the ϵ approximate problem which connect (t_o, x_o) with a point on S and lie in $\mathcal{A}^-(\epsilon, S)$. Then Ω^ϵ is not empty since $(t_o, x_o) \in \mathcal{A}^-(\epsilon, S)$ and φ^ϵ a characteristic implies $\{(t, \varphi^\epsilon(t)): t_o \leq t \leq t_\epsilon\}$ is in $\mathcal{A}^-(\epsilon, S)$. If $v^*(t, \varphi^\epsilon(t), V_x^\epsilon(t, \varphi^\epsilon(t))$ is the only function (to within a set of zero measure) in Ω^ϵ, the result is trivially true. If this is not the case let $v = v(t)$ be any function in Ω^ϵ differing from $v^*(t, \varphi^\epsilon(t), V_x^\epsilon(t, \varphi^\epsilon(t)))$ on a set Λ of positive measure. Let $\varphi(\cdot\,; v)$ be the corresponding solution of the approximate system and t_2 the first time such that $(t_2, \varphi(t_2; v)) \in S$. $(t_2 > t_o)$. We must show $t_\epsilon < t_2$.

Calculating

$$\frac{d}{dt} V^\epsilon(t, \varphi(t; v)) - 1 \equiv V_t^\epsilon(t, \varphi(t;v)) + V_x^\epsilon(t, \varphi(t;v)) \cdot h^\epsilon(t, \varphi(t;v), v(t)) - 1 \leq 0$$

for all t and strictly less than zero for $t \in \Lambda$, implying $V^\epsilon(t_2, \varphi(t_2; v)) - V^\epsilon(t_o, x_o) < t_2 - t_o$. But $V^\epsilon(t_2\ \varphi(t_2; v)) = 0$ since $(t_2, \varphi(t_2; v)) \in S$, yielding $-V^\epsilon(t_o, x_o) < t_2 - t_o$. Similarly $\frac{d}{dt} V^\epsilon(t, \varphi^\epsilon(t)) - 1 \equiv 0$ implying $- V^\epsilon(t_o, x_o) = t_\epsilon - t_o$. Combining the last two inequalities gives $t_\epsilon < - t_2$ as was to be shown. ∎

THE CONSTRUCTION OF APPROXIMATING PROBLEMS WHEN THE CONTROL APPEARS LINEARLY.

Theorem III.4 gives conditions for the existence of an ϵ equivalent approximate problem which has the unit ball B^n as the set of values which the control can assume. However, the functional form of the approximating system is allowed to vary with ϵ.

In this section we consider a system of the form

$$\text{(3-13)} \qquad \dot{x}(t) = g(t, x(t)) + H(t, x(t))\, u(t),$$

$u(t) \in U$, a compact convex set in E^r with $1 \leq r \leq n$; H an nxr matrix valued C^2 function; while g is a C^2, n vector valued function. For such systems it is possible to provide a simple construction for ϵ approximate problems.

Since, for the approximate problem, one desires $R(t, x, \epsilon)$ to be strictly convex and lemma III.1 shows this implies non void interior, one is led to extend H to an nxn matrix valued function and approximate the control set by a compact set $V(\epsilon)$ which contains U. Furthermore, $V(\epsilon)$ should have a non-void n dimensional interior, a smooth boundary with positive Gaussian curvature, and be such that in the Hausdorff metric topology, $\lim_{\epsilon \to 0} V(\epsilon) = U$.

The method of construction and the application to approximating problems will be demonstrated in a two dimensional example; its generalization to higher dimensions being immediate.

Example III.1 (Bushaw control problem).

Consider the time optimal problem for the system

$$\text{(3-14)} \qquad \begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -x_1 + u \end{aligned}$$

with arbitrary initial data $x(0) = x_0$, and target $S = \{(t, x_1, x_2): x_1 = 0, x_2 = 0\}$. The control u is to satisfy $-1 \leq u(t) \leq 1$, i.e., $U = [-1,1]$.

As an ϵ approximate problem we take the system

$$\text{(3-15)} \qquad \dot{x}_1 = x_2 + v_1$$
$$\dot{x}_2 = -x_1 + v_2$$

with the same initial data and target, but with $V(\epsilon) = \left\{ v \in E^2 : v_1{}^2 + \epsilon^2 v_2{}^2 \leq \epsilon^2 \right\}$,

i.e., an ellipse with semi major axis 1 and semi minor axis ϵ. Thus in the Hausdorff metric topology $\lim_{\epsilon \to 0} V(\epsilon) = U$, and $\partial R(t, x, \epsilon)$ is smooth with positive Gaussian curvature. From the Hamilton-Jacobi theory

$$H(t, x, p, v, \epsilon) \equiv p_1 x_2 + p_1 v_1 - p_2 x_1 + p_2 v_2 - 1.$$

Using lemma III.3 one computes

$$v^*(t, x, p^* \equiv (\epsilon^2 p_1 \left[\epsilon^2 p_1{}^2 + p_2{}^2\right]^{-1/2}, \ p_2 \left[\epsilon^2 p_1{}^2 + p_2{}^2\right]^{-1/2}$$

from which it follows that

$$H^*(t, x, p, \epsilon) \equiv p_1 x_2 - p_2 x_1 + \left[p_1{}^2 \epsilon^2 + p_2{}^2\right]^{-1/2} - 1.$$

The associated Hamilton-Jacobi equation is

$$\text{(3-16)} \quad V_t(t,x) + x_2 V_{x_1}(t,x) - x_1 V_{x_2}(t,x) + \left[\epsilon^2 V_{x_1}^2(t,x) + V_{x_2}^2(t,x)\right]^{1/2} - 1 = 0.$$

Since the independent variables appear linearly, while the dependent variable has derivatives which appear non-linearly, the Legendre contact transformation is suggested. Let $V(t,x) = W(t,p) - p \cdot x$. Then $V_t = W_t$, $V_x = -p$, $W_p = x$ and the transformed equation is

$$W_t(t,p) - p_1 W_{p_2}(t,p) + p_2 W_{p_1}(t,p) + \left[\epsilon^2 p_1{}^2 + p_2{}^2\right]^{1/2} - 1 = 0.$$

The characteristic equations associated with this linear partial differential equation are $t'(\tau) = 1$, $p_1'(\tau) = p_2(\tau)$, $p_2'(\tau) = -p_1(\tau)$, yielding solutions: $t = \gamma + \tau$, $p_1 = \alpha \sin(\tau + \beta)$, $p_2 = \alpha \cos(\tau + \beta)$ with α, β, γ arbitrary constants. Then $\frac{d}{dt} W(t(\tau), p(\tau)) = 1 - \left[\epsilon^2 p_1^2(\tau) + p_2^2(\tau)\right]^{1/2}$

which, after a slight calculation, gives

$$W(t,p_1,p_2;\delta,\gamma) = t-\gamma+\delta+\int_0^{(\gamma-t)} \left[\epsilon^2(p_2 \sin\tau + p_1 \cos\tau)^2 + (p_2 \cos\tau - p_1 \sin\tau)^2\right]^{1/2} dt.$$

For a time optimal problem with autonomous system equations and target a a point in state space, the constant δ is inconsequential. We consider $\delta = 0$ and omit further reference to it.

By virtue of the transformation, solution trajectories to the system (3-15) with $v = v^*(t, x, p)$ are given by $x(t;\alpha,\beta,\gamma) = W_p(t,p(t;\alpha,\beta);\gamma)$ or specifically

$$x_1(t;\alpha,\beta,\gamma) = \int_0^{(\gamma-t)} \frac{\alpha\epsilon^2 \sin(2\tau+\beta)\cos\tau - \alpha\cos(2\tau+\beta)\sin\tau}{\left[\epsilon^2\alpha^2 \sin^2(2\tau+\beta) + \alpha^2\cos^2(2\tau+\beta)\right]^{1/2}}\, d\tau$$

$$x_2(t;\alpha,\beta,\gamma) = \int_0^{(\gamma-t)} \frac{\alpha\,\epsilon^2 \sin(2\tau+\beta)\sin\tau + \alpha\cos(2\tau+\beta)\cos\tau}{\left[\epsilon^2\alpha^2 \sin^2(2\tau+\beta) + \alpha^2\cos^2(2\tau+\beta)\right]^{1/2}}\, d\tau$$

These formulas can be interpreted as follows. If we choose $\gamma > 0$ and $t = 0$, $\left\{x(0;\alpha,\beta,0) : (\alpha_1\beta) \in E^2\right\}$ gives the set of initial points x_o from which the origin can be reached in time γ by trajectories which satisfy (3-15) with $v = v^*(t, x, p)$. In particular, it can be shown (via the theory of homogeneous contact transformations) that the jacobian determinant

$\dfrac{\partial(x_1, x_2)}{\partial(\alpha, \beta)}$ is zero, and in this case the set of initial points forms a closed curve in E^2 for each $\gamma > 0$.

To generate a field of extremals (it is to be cautioned that the term extremal is to be taken in the sense of the classical calculus of variations; i.e., not necessarily to infer optimality) choose $\gamma = 0$ and replace t with -t in (3-17). For each choice of α, β one obtains an extremal which is at the origin at time zero. Varying α, β now gives a field of extremals.

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LAUNCH TRAJECTORIES WITH INEQUALITY CONSTRAINTS

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SUMMARY

A method is presented for finding an extremal solution of launch trajectories involving certain inequality path constraints. The method of solution is an extension of a technique outlined by Denbow. Denbow has formulated the problem considering two or more intersecting admissible arcs; and by a suitable transformation of the independent variable, these admissible arcs may be transformed into a single admissible arc in the problem of Bolza. The transformation leads to additional transversality conditions at the intersection of the arcs in addition to the usual set of tranversality conditions at the final time.

The problem considered in this report is the maximization of final weight and is solved numerically by the Mayer formulation of the calculus of variations. The problem is represented by three arcs - two unconstrained arcs and one constrained arc due to the inequality path constraint.

The optimum control variables are obtained from the Euler-Lagrange equations while the trajectory is moving along the unconstrained arcs. While on the constrained arc of the trajectory, the control variables are determined from the constraint equation.

TABLE OF CONTENTS

Section I

INTRODUCTION

A method is presented for finding an extremal solution of exit phase trajectories involving certain inequality path constraints. The example problem considered involves determining the optimum control program to extremalize any desired functional of the coordinates (final vehicle mass). The method of solution employed is an extension of the classical method of the variational calculus as outlined by Denbow. Denbow has shown that two or more admissible arcs can be transformed, by a suitable transformation of the independent variable, into a single admissible arc in the problem of Bolza. The problem of concern has three admissible arcs as shown in Figure 1. The results of Denbow's work indicates that at the intersection of these arcs certain transversality conditions must be satisfied in order to insure an extrema over the trajectory.

The matrix form of the transversality equation is used to determine the necessary transversality conditions. In addition to the final transversality conditions, certain other transversality conditions must be determined at the intermediate points (corners of arcs) along the trajectory.

The equations of motion are written for a point mass in three-dimensional space in an inverse-square gravitational field. The optimum control variables are obtained from the Euler-Lagrange equations on the unconstrained arcs of the trajectory and from the constraint equation while on the constrained arc of the trajectory.

This report presents the solution to the problem as a result of the application of Denbow's results.

Section 2

MATHEMATICAL FORMULATION

2.1 Statement of Problem

The problem may be stated as maximizing the vehicle mass to some given set of end conditions through first stage flight. In addition, certain inequality path constaints are imposed on the trajectory. The statement "first stage flight" entails the meaning of flight within the sensible atmosphere. The particular constraint considered in this report is the product of the angle of attack of the vehicle and its dynamic pressure. This type path constraint gives an indication of the aerodynamic loads the vehicle will encounter; hence, this constraint may be used to control the structural bending.

The method of solution chosen is an extension of a technique outlined by Denbow, Reference 1. This method is essentially the classical calculus of variations technique developed by Bliss (Reference 2).

Denbow has shown that the original problem may be transformed to an equivalent problem of Bolza (the Mayer formulation is used) by a suitable transformation of the independent variable. This transformation takes the three subarcs depicted in Figure 1 for our problem and combines them into a single admissible arc. The results of Denbow's paper applied to the stated problem lead to additional transversality conditions at the variable intermediate points t_1 and t_2.

2.2 Equations of Motion

The equations of motion are written for a space vehicle traveling in an inverse-square force field. Three degrees of freedom are used to describe the motion about a non-rotating spheroid. The trajectory variables are defined as:

$$\bar{X} = \left[\begin{array}{c} m \\ \bar{R}_I \\ \bar{V}_I \end{array}\right] \begin{array}{l} \text{Vehicle Mass} \\ \text{Inertial Frame Position} \\ \text{Inertial Frame Velocity} \end{array} \qquad (2.1)$$

and the control variables as:

$$\bar{\mu} = \left[\begin{array}{c} \xi \\ \chi \\ \tau \end{array}\right] \begin{array}{l} \text{Inertial Roll Attitude of Thrust Vector} \\ \text{Inertial Pitch Attitude of Thrust Vector} \\ \text{Inertial Yaw Attitude of Thrust Vector} \end{array} \qquad (2.2)$$

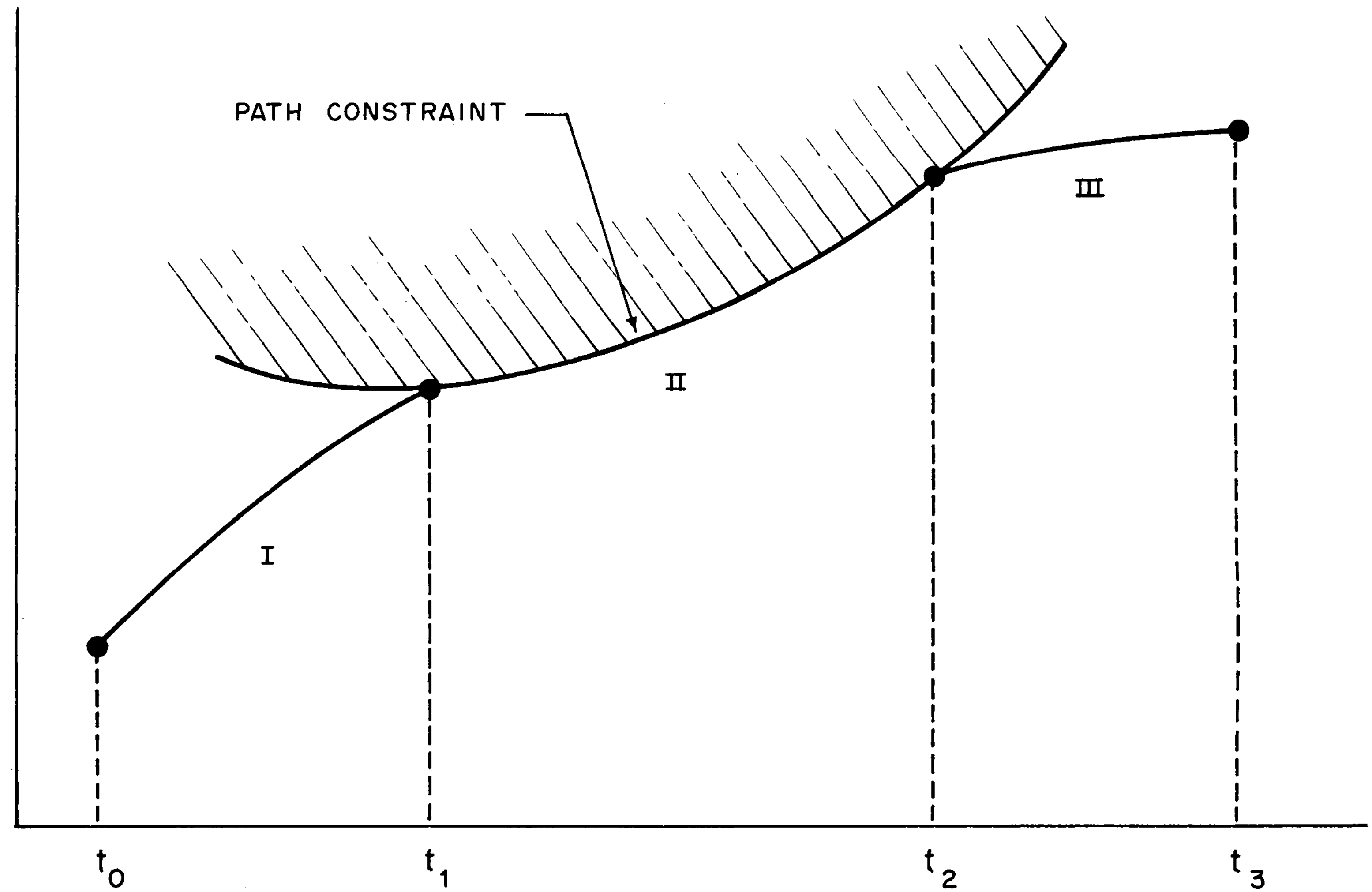

FIGURE 1

Time, t, is taken as the independent variable. The mass flow rate is constant and the thrust vector, T, is assumed to be directed along the longitudinal axis of the missile. In addition, roll effects are ignored throughout the analysis.

The powered flight equations are then

$$\dot{\bar{X}} = \begin{bmatrix} \dot{m} \\ \bar{V}_I \\ \bar{A}_I + \bar{G}_I \end{bmatrix} \tag{2.3}$$

where $\bar{A}_I$, is the non-gravitational acceleration vector in the inertial frame. The thrust and drag are given by

$$A_{I_X} = \frac{F}{m} c\chi c\tau$$

$$A_{I_Y} = \frac{F}{m} c\chi s\tau$$

$$A_{I_Z} = -\frac{F}{m} s\chi$$

(Note: $c\chi \equiv \cos\chi$, $s\chi \equiv \sin\chi$, etc.)

where

$$F = T_a - F_X$$

T_a = thrust at altitude along X_M-axis

F_X = drag along X_M axis

and $\bar{G}_I$ is the gravity vector in the inertial frame. The aerodynamic lift force has been neglected in this formulation. The addition of lift would add excessive complication to the control equations and require supplemental iteration to arrive at a solution.

For consistency in nomenclature, the differential equations of motion for our problem will be written in the following vector notation.

$$\bar{\Phi}_\beta = \dot{\bar{X}}_\beta - \bar{f}_\beta(\bar{X}, \bar{u}, t), \quad (\beta = 1, 2, \ldots, 7) \tag{2.4}$$

where $\dot{\bar{X}}_\beta$ is given in equation(2.3).

and

$$\bar{f}_\beta = \begin{bmatrix} f_1 \\ f_2 \\ \vdots \\ f_\eta \end{bmatrix}$$

is a β -vector of known functions of $\bar{X}(t)$, $\bar{\mu}(t)$, and t , assumed everywhere differentiable with respect to $\bar{X}$ and $\bar{\mu}$.

2.3 Mayer Formulation of Problem

The functional to be maximized, Equation (2.7) is written directly in terms of the boundary conditions. Since the functional does not involve an integral (as in the problem of Bolza), the problem can be stated using the Mayer formulation of the classical method of the calculus of variations. The Mayer formulation of the problem may be written explicitly as: In a class of admissible arcs, $\bar{X}_i(t)$, satisfying the differential Equation (2.4)

$$\bar{\phi}_\beta = \dot{\bar{X}}_\beta - \bar{f}_\beta(\bar{X}, \bar{\mu}, t) = 0$$

and end conditions of the form

$$\eta_\mu = \eta_\mu(\bar{X}_f, \bar{\mu}_f) \tag{2.5}$$

as well as the inequality path constraint

$$\psi = \alpha_T \bar{q} - \left|\alpha_T \bar{q}\right|_{max} \leq 0 \tag{2.6}$$

where α_T is the total angle of attack and $\bar{q}$ is defined as the dynamic pressure, find the specific arc that minimizes

$$J = g(\bar{X}, t)\Big|_{t_0}^{t_f} = -(m_f - m_0) \tag{2.7}$$

It may be recalled that the problem of maximizing final vehicle mass (m_f) is identical with that of minimizing $(-m_f)$.

The problem is shown schematically in Figure 1. The intermediate points t_1 and t_2 , vary with each independent trajectory. The entry corner, t_1 , (point where the trajectory goes from unconstrained arc onto a constrained arc) is defined to be that time at which the product $\alpha \bar{q}$ just equals to the maximum allowable value. The exit corner (point where the solution goes from the constrained arc onto an unconstrained arc) is denoted by t_2 . The criteria for determining this time, t_2 , will be given in the Section on Control Equations.

By the calculus of variations technique, an extremal solution must satisfy the following Euler-Lagrange equations given explicitly as

$$\begin{aligned} \dot{\bar{\lambda}}(t) + \bar{\lambda}^T \frac{\partial \bar{f}}{\partial \bar{X}} &= 0 \\ \bar{\lambda}(t) \frac{\partial \bar{f}}{\partial \bar{\mu}} &= 0 \end{aligned} \tag{2.8}$$

where

$$\bar{\lambda}(t) = \begin{bmatrix} \lambda_1(t) \\ \vdots \\ \lambda_7(t) \end{bmatrix} \tag{2.9}$$

is a vector of Lagrange multiplier functions and $\bar{\lambda}^T(t)$ is the transpose of $\bar{\lambda}(t)$.

The above Euler-Lagrange equations were derived explicitly through use of the augmented function

$$\begin{aligned} \mathcal{F} &= \bar{\lambda}_\beta \bar{\Phi}_\beta \\ &= \lambda_1(\dot{X}_1 - \dot{m}) + \lambda_2(\dot{X}_2 - V_X) + \lambda_3(\dot{X}_3 - V_Y) + \lambda_4(\dot{X}_4 - V_Z) \\ &\quad + \lambda_5(\dot{X}_5 - A_X) + \lambda_6(\dot{X}_6 - A_Y) + \lambda_7(\dot{X}_7 - A_Z). \end{aligned} \tag{2.10}$$

2.4 Optimum Control Equations

The optimum control variables, while on the unconstrained arcs (I and III), are found by solving the second set of Euler-Lagrange Equations (2.8) explicitly for χ and τ.

The optimum control variables, while the trajectory is on the unconstrained arcs, are then found to be

$$\bar{u} = \begin{bmatrix} \xi \\ \chi \\ \tau \end{bmatrix} = \begin{bmatrix} 0 \\ \tan^{-1}\left(\dfrac{-\lambda_7}{\lambda_5 c\tau + \lambda_6 s\tau}\right) \\ \tan^{-1}\left(\dfrac{\lambda_6}{\lambda_5}\right) \end{bmatrix} \tag{2.11}$$

When the vehicle is moving along the constrained arc II, one of the control variables may be obtained from the constraint Equation (2.6). This reduces the degree of optimality by one for flight along the constraining arc. (Although strictly arbitrary, it was decided to solve the constraint equation directly for χ).

The pitch command angle, χ, was solved from the constraint equation as follows:

First, it was necessary to have the constraint equation as an explicit function of the control variables. To do this, the constraint equation

$$\psi = \alpha_T \bar{q} - L_M \le 0 \qquad L_M = |\alpha \bar{q}|_{MAX.}$$

was redefined as

$$\psi \doteq \sin \alpha_T \bar{q} - L_M \leq 0 \tag{2.12}$$

without loss of generality, since for small allowable α_T

$$\sin \alpha_T = \alpha_T$$

Thus, the modified constaint equation is essentially unchanged. While on the constraining portion of the trajectory,

$$\sin \alpha_T \bar{q} = L_M$$

and a fundamental trigometric identity enables us to write

$$\sin \alpha_T = \sqrt{1 - \cos^2 \alpha_T} = L_M / \bar{q}$$

or

$$\cos^2 \alpha_T = 1 - L_M^2 / \bar{q}^2$$

The total angle of attack, α_T, is defined as the angle between the thrust and velocity vectors, which may be written vectorially as

$$\cos \alpha_T = \frac{\bar{V} \odot \bar{X}_M}{|\bar{V}| \, |\bar{X}_M|} ;$$

$\odot$ is the vector dot product

$|\ |$ is the absolute value

or explicitly in terms of the control variables this becomes

$$\cos \alpha_T = \frac{\dot{X} c\chi c\tau + \dot{Y} c\chi s\tau - \dot{Z} s\chi}{|\bar{V}|}$$

By substituting the above equation into the modified constraint equation (2.12) and collecting terms, a quadratic in $\sin \chi$ may be obtained,

$$s^2\chi + \left[\frac{2K^* \dot{Z}}{\dot{Z}^2 + (\dot{X} c\tau + \dot{Y} s\tau)^2} \right] s\chi + \left[\frac{K^{*2} - (\dot{X} c\tau + \dot{Y} s\tau)^2}{\dot{Z}^2 + (\dot{X} c\tau + \dot{Y} s\tau)^2} \right] = 0 \tag{2.13}$$

where

$$K^* = \left(\sqrt{1 - \frac{L_M^2}{\bar{q}^2}} \right) |\bar{V}|$$

Hence, the pitch command angle, χ, may be determined as a function of τ on the constraint.

To solve for the optimum yaw command angle, τ, on the constraint, it is necessary to modify the second set of Euler-Lagrange Equations (2.8). The above Equation (2.13) is solved for χ in terms of τ and substituted into these Euler-Lagrange equations. When on the constraint, these modified Euler-Lagrange equations may be solved explicitly to give

$$\lambda_5\left(\frac{\partial c\chi}{\partial \tau} c\tau - c\chi c\tau\right) + \lambda_6\left(c\chi c\tau + \frac{\partial c\chi}{\partial \tau} s\tau\right) - \lambda_7 \frac{\partial s\chi}{\partial \tau} = 0 \qquad (2.14)$$

The two equations (2.13 and 2.14) may be solved by a Newton-Raphson iterative scheme to obtain the control variables χ and τ on the constrained arc.

As described in the Section on Mayer Formulation of Problem, the entry time, t_1, is that time at which $\alpha \bar{q}$ equals the constraint limit, L_M. Thus, starting at time t_1, χ will be determined from the constraint equation. The criteria for determing the exit time, t_2, will be established as follows: The exit corner time will be defined as that time when chi solved from the constraint equation is equal to chi if solved from the Euler-Lagrange equation. Then, on the unconstrained arc III χ and τ are solved from the Euler-Lagrange equations (see Equation (2.11)).

2.5 Transversality Conditions

A method for determining transversality conditions was suggested in Hunt's paper (Reference 3) and involved construction of the matrix shown in Figure 2. The matrix form of the transversality conditions for the problem of Mayer was formulated as follows

(a) Top Row

  (A) Independent Variable (t_0)

  (B) The Dependent Variables ($\bar{X}_i$) at t_0

  (C) Next (A) and (B) are repeated for t_1, t_2, and t_3

(b) First Column

  (A) Initial Conditions

  (B) Boundary Conditions

  (C) $g(\bar{X}_f, t_f)$ of Mayer problem

  (D) Special Case - Partial Derivatives of augmented function, $\mathcal{F}$, with respect to $\bar{X}_i$.

Q + 1 rows

(c) The partial derivatives of the Q column elements with respect to the row elements were calculated to complete the matrix. The blanks on Figure 2 indicate zero elements. The transversality conditions are now determined from the augmented matrix by the relationship that all determinants of order Q + 1 must be equal to zero.

	t_0	m_0	X_0	Y_0	Z_0	$\dot{X}_0$	$\dot{Y}_0$	$\dot{Z}_0$	χ_0	τ_0
t_0	1									
m_0		1								
X_0			1							
Y_0				1						
Z_0					1					
$\dot{X}_0$						1				
$\dot{Y}_0$							1			
$\dot{Z}_0$								1		
$\psi=\sin\alpha\bar{q}-L_M\leq 0$										
$\eta_1=h_f-h_{stop}=0$										
$\eta_2=\bar{V}_f-\bar{V}_{stop}=0$										
$-m_f$										
$\bar{\lambda}$	G_1	λ_{10}	λ_{20}	λ_{30}	λ_{40}	λ_{50}	λ_{60}	λ_{70}	0	0

$i = 1, 2$

	t_i	m_i	X_i	Y_i	Z_i	$\dot{X}_i$	$\dot{Y}_i$	$\dot{Z}_i$	χ_i	τ_i
t_0 … $\dot{Z}_0$										
$\psi=\sin\alpha\bar{q}-L_M\leq 0$	0	0	ψ_{X_i}	ψ_{Y_i}	ψ_{Z_i}	$\psi_{\dot{X}_i}$	$\psi_{\dot{Y}_i}$	$\psi_{\dot{Z}_i}$	ψ_{χ_i}	ψ_{τ_i}
$\eta_1=h_f-h_{stop}=0$										
$\eta_2=\bar{V}_f-\bar{V}_{stop}=0$										
$-m_f$										
$\bar{\lambda}$	$G_1^- - G_1^+$	$\lambda_{1i}^- - \lambda_{1i}^+$	$\lambda_{2i}^- - \lambda_{2i}^+$	$\lambda_{3i}^- - \lambda_{3i}^+$	$\lambda_{4i}^- - \lambda_{4i}^+$	$\lambda_{5i}^- - \lambda_{5i}^+$	$\lambda_{6i}^- - \lambda_{6i}^+$	$\lambda_{7i}^- - \lambda_{7i}^+$	0	0

	t_3	m_3	X_3	Y_3	Z_3	$\dot{X}_3$	$\dot{Y}_3$	$\dot{Z}_3$	χ_3	τ_3
t_0 … $\dot{Z}_0$										
$\psi=\sin\alpha\bar{q}-L_M\leq 0$										
$\eta_1=h_f-h_{stop}=0$	0	0	$\frac{X_3}{R'}$	$\frac{Y_3}{R'}$	$\frac{Z_3}{R'}$	0	0	0	0	0
$\eta_2=\bar{V}_f-\bar{V}_{stop}=0$	0	0	0	0	0	$\frac{\dot{X}_3}{V}$	$\frac{\dot{Y}_3}{V}$	$\frac{\dot{Z}_3}{V}$	0	0
$-m_f$	0	-1	0	0	0	0	0	0	0	0
$\bar{\lambda}$	G_3	λ_{13}	λ_{23}	λ_{33}	λ_{43}	λ_{53}	λ_{63}	λ_{73}	0	0

Figure 2 - Matrix Form of Transversality Conditions

As the entry point, t_1, and exit point, t_2, the evaluation of the determinants results in five independent transversality conditions relating the changes in the λ's (denoted by $\Delta\bar{\lambda}$) across the constraint boundaries. For the boundary points t_1 and t_2, the TVC's are

$$\begin{aligned}
&\psi_X \Delta\lambda_3 - \psi_Y \Delta\lambda_2 = 0 \\
&\psi_X \Delta\lambda_4 - \psi_Z \Delta\lambda_2 = 0 \\
&\psi_X \Delta\lambda_5 - \psi_{\dot{X}} \Delta\lambda_2 = 0 \\
&\psi_X \Delta\lambda_6 - \psi_{\dot{Y}} \Delta\lambda_2 = 0 \\
&\psi_{\dot{X}} \Delta\lambda_7 - \psi_{\dot{Z}} \Delta\lambda_5 = 0
\end{aligned} \tag{2.15}$$

The Lagrange multiplier λ_1, associated with the mass doesn't enter into the equations and was omitted from the analysis. Another λ may be set equal to one due to homogeneity of Euler-Lagrange equation. This leaves five λ's (λ_3 to λ_7) to be determined. Investigating the intermediate points, we find that an additional transversality equation is available at these boundary points, t_1 and t_2.

From the augmented matrix, the following relations must hold:

$$\psi_\chi \Delta\bar{\lambda} = 0 \qquad\qquad \psi: \sin \alpha_T \bar{q} - L_M = 0$$

$$\psi_\tau \Delta\bar{\lambda} = 0$$

Since the $\Delta\bar{\lambda}$'s are not zero, these equations stipulate that

$$\psi_\chi = \psi_\tau = 0$$

From the preceding section, it was seen that the constraint equation may be written as:

$$\psi = \dot{X} c\chi c\tau + \dot{Y} c\chi s\tau - \dot{Z} s\chi - K^* = 0$$

where

$$K^* = \left(\sqrt{1 - \frac{L_M^2}{\bar{q}}}\right) |\bar{V}|$$

This equations yields

$$\begin{aligned}
\psi_\chi &= -\dot{X} s\chi c\tau - \dot{Y} s\chi s\tau - \dot{Z} c\chi = 0 \\
\psi_\tau &= -\dot{X} c\chi s\tau + \dot{Y} c\chi c\tau = 0
\end{aligned} \tag{2.16}$$

Simple manipulations of the optimum control Equations (2.11) give the following relations:

$$\sin \tau = \frac{\lambda_6}{\sqrt{\lambda_5^2 + \lambda_6^2}} \quad ; \quad \cos \tau = \frac{\lambda_5}{\sqrt{\lambda_5^2 + \lambda_6^2}}$$

Therefore,

$$\tan \chi = \frac{-\lambda_7}{\sqrt{\lambda_5^2 + \lambda_6^2}}$$

and

$$\sin\chi = \frac{-\lambda_7}{|\bar{\lambda}|} \quad , \qquad \cos\chi = \frac{\sqrt{\lambda_5^2 + \lambda_6^2}}{|\bar{\lambda}|}$$

where

$$|\bar{\lambda}| = \sqrt{\lambda_5^2 + \lambda_6^2 + \lambda_7^2}$$

This gives

$$\sin\chi \sin\tau = -\frac{\lambda_6 \lambda_7}{|\bar{\lambda}|\sqrt{\lambda_5^2 + \lambda_6^2}} \quad , \quad \sin\chi\cos\tau = -\frac{\lambda_5 \lambda_7}{|\bar{\lambda}|\sqrt{\lambda_5^2 + \lambda_6^2}}$$

$$\cos\chi\cos\tau = \frac{\lambda_5}{|\bar{\lambda}|} \quad , \quad \cos\chi\sin\tau = \frac{\lambda_7}{|\bar{\lambda}|}$$

Substituting the necessary relations into Equation (2.16) and solving simultaneously, we obtain the following transversality condition valid on the unconstrained side of points t_1, and t_2.

$$\lambda_7 = \lambda_5\left(\frac{\dot{Z}}{\dot{X}}\right) \tag{2.17}$$

It is interesting to note that this equation enables us to determine $\Delta\lambda_5$ or $\Delta\lambda_7$ at point t_2 from which the remaining $\Delta\bar{\lambda}$'s may be determined from Equation (2.15). Thus, all the TVC's at point t_2 are trivially satisfied.

However, at the entry point, t_1, the above TVC (Equation 2.17) is valid on the left unconstrained side of the point. Therefore, we must isolate on one of the $\Delta\bar{\lambda}$'s at point t_1, from which the remaining $\Delta\bar{\lambda}$'s may be determined from Equation (2.15). Equation (2.17) then remains as a transversality equation that must be satisfied at the entry point, t_1.

A specified altitude and velocity was selected as end conditions for this particular example problem. Velocity will be the "cutoff" criterion leaving altitude as a transversality condition to be satisfied at the final time, t_3. This set of end conditions gave simplified final TVC's focusing attention to the intermediate TVC's at t_1 and t_2. The remaining final TVC's are obtained from the Transversality Matrix. The final TVC's are given in the following equations.

$$\begin{aligned} TVC(1) &= h_f - h_{stop} = 0 \\ TVC(2) &= \lambda_3 X - \lambda_2 Y = 0 \\ TVC(3) &= \lambda_4 X - \lambda_2 Z = 0 \\ TVC(4) &= \lambda_6 \dot{X} - \lambda_5 \dot{Y} = 0 \\ TVC(5) &= \lambda_7 \dot{Y} - \lambda_6 \dot{Z} = 0 \end{aligned} \tag{2.18}$$

Summarizing the results above, we find that there are five transversality conditions to be satisfied at the final time, t_3, and one remaining transversality condition to be satisfied at the entry corner, t_1. Thus it is necessary to have six adjoint variables that we can vary in order to satisfy these six transversality conditions.

The total number of initial adjoint variables available at t_0 is seven. However, one of these, namely λ_1, is associated with the vehicle mass and does not appear in the transversality conditions and is omitted. One other may be set equal to one due to the homogeneity of the Euler-Lagrange equations. This gives a total of five initial $\bar{\lambda}$'s that must be chosen. The sixth adjoint variable free to choose is one of the $\Delta\bar{\lambda}$'s across the entry corner, t_1 .

A systematic search routine is employed to determine the values of the five initial $\bar{\lambda}$'s and the $\Delta\bar{\lambda}$'s across the exit point in order to satisfy the transversality conditions. By satisfying these transversality conditions, an optimum trajectory for the entire first stage flight results.

Section 3

REFERENCES

1. Denbow, Carl Herbert, A Generalized Form of the Problem of Bolza, University of Chicago, October 1937.

2. Bliss, Gilbert A., Lectures on the Calculus of Variations, University of Chicago Press, 1959.

3. Hunt, R. N., A Generalized Bolza-Mayer Problem with Discontinuous Solutions and Variable Intermediate Points, (notes received from NASA/MSFC Aeroballistics Division, March 1964.)

Appendix A

REFERENCE FRAMES

A description of the four coordinate frames used in the analysis is stated below. The relationships among these reference frames are shown on Figure 1A. It may be noted that all reference frames are right-hand coordinate systems.

1. R-Frame This Reference-frame has its X_R -axis through the prime meridian at the time of launch and its Z_R -axis through the North Pole. The Y_R -axis is oriented to form a right-hand coordinate system.

2. I-Frame This is an Inertial-frame centered at the launch site with the X_I -axis in the negative direction of the gravity vector, $\bar{G}$, and Z_I -axis at a given azimuth from North. The orientation of the I-frame is found from the R-frame by the following sequence of rotation:

   (a) Rotation about the Z_R - axis by the longitude of the launch point,

   (b) Rotation about the new Y_R' - axis in the negative direction by the plumbline latitude, -

   (c) Rotation about the new X_R'' - axis in the negative direction by the firing azimuth, $-A_{ZL}$;

The azimuth is measured in the plane normal to the local geodetic or plumbline direction.

3. L-Frame This is the Local Horizontal (octangent)-frame which is normal to the local geodetic or plumbline direction. The L-frame is transformed from the R-frame by the following rotations:

   (a) Rotation about the Z_R - axis by the instantaneous longitude, ϕ .

   (b) Rotation about the new Y_R' - axis by $(-\psi-\frac{\pi}{2})$, where ψ is the instantaneous latitude.

4. M-Frame The origin of the Missile-frame is at the center of gravity of the missile with the X_M - axis forward along the longitudinal axis of the missile. The negative Z_M - axis is in the direction of aerodynamic lift. The transformation is obtained from the I-frame by the Euler angle sequence of rotation:

   (a) Rotation about the X_I - axis by the inner gimbal angle, ξ .

   (b) Rotation about the new Z_I' - axis by the middle gimbal angle, τ .

   (c) Rotation about the new Y" - axis by the outer gimbal angle, χ.

The three gimbal angles ξ, τ, and χ are the command attitude angles. They describe the orientation of the thrust vector in the I-frame.

The transformation matrices from one frame to another are given below. These transformations are formed by successive rotations of the Euler angles described in the preceding paragraphs. The transformation from R-frame to I-frame, T_{R2I}, is time invariant (remains fixed from launch).

$$\left[T_{R2I}\right] = \begin{bmatrix} CA_{ZLO}C\psi_o & -CA_{ZLO}S\psi_o C\phi_o - SA_{ZLO}S\phi_o & -CA_{ZLO}S\psi_o S\phi_o + SA_{ZLO}C\phi_o \\ S\psi_o & C\psi_o C\phi_o & C\psi_o S\phi_o \\ -SA_{ZLO}C\psi_o & SA_{ZLO}S\psi_o C\phi_o - CA_{ZLO}S\phi_o & SA_{ZLO}S\psi_o S\phi_o + CA_{ZLO}C\phi_o \end{bmatrix} \quad (A.1)$$

$$\left[T_{R2L}\right] = \begin{bmatrix} -S\psi C\phi & -S\psi S\phi & C\psi \\ -S\phi & C\phi & 0 \\ -C\psi C\phi & -C\psi S\phi & -S\psi \end{bmatrix} \quad (A.2)$$

$$\left[T_{I2M}\right] = \begin{bmatrix} C\chi C\tau & C\chi S\tau C\xi + S\chi S\xi & C\chi S\tau S\xi - S\chi C\xi \\ -S\tau & C\tau C\xi & C\tau S\xi \\ S\chi C\tau & S\chi S\tau C\xi - C\chi S\xi & S\chi S\tau S\xi + C\chi C\xi \end{bmatrix} \quad (A.3)$$

$$\left[T_{L2M}\right] = \left[T_{I2M}\right] \cdot \left[T_{R2I}\right] \cdot \left[T_{L2R}\right] \quad (A.4)$$

The transformations T_{R2M}, and T_{I2L} are determined in a similar fashion. The inverse (transpose) of each matrix is

$$\left[T_{B2A}\right] = \left[T_{A2B}\right]^{-1} = \left[T_{A2B}\right]^{T}$$

since all of the transformations are orthogonal.

SPHERICAL RELATIONS

$$R' = \sqrt{X_R^2 + Y_R^2 + Z_R^2}$$

$$\phi = \tan^{-1}\left(\frac{Y_R}{X_R}\right)$$

$$\psi = \tan^{-1}\left(\frac{Z_R}{\sqrt{X_R^2 + Y_R^2}}\right)$$

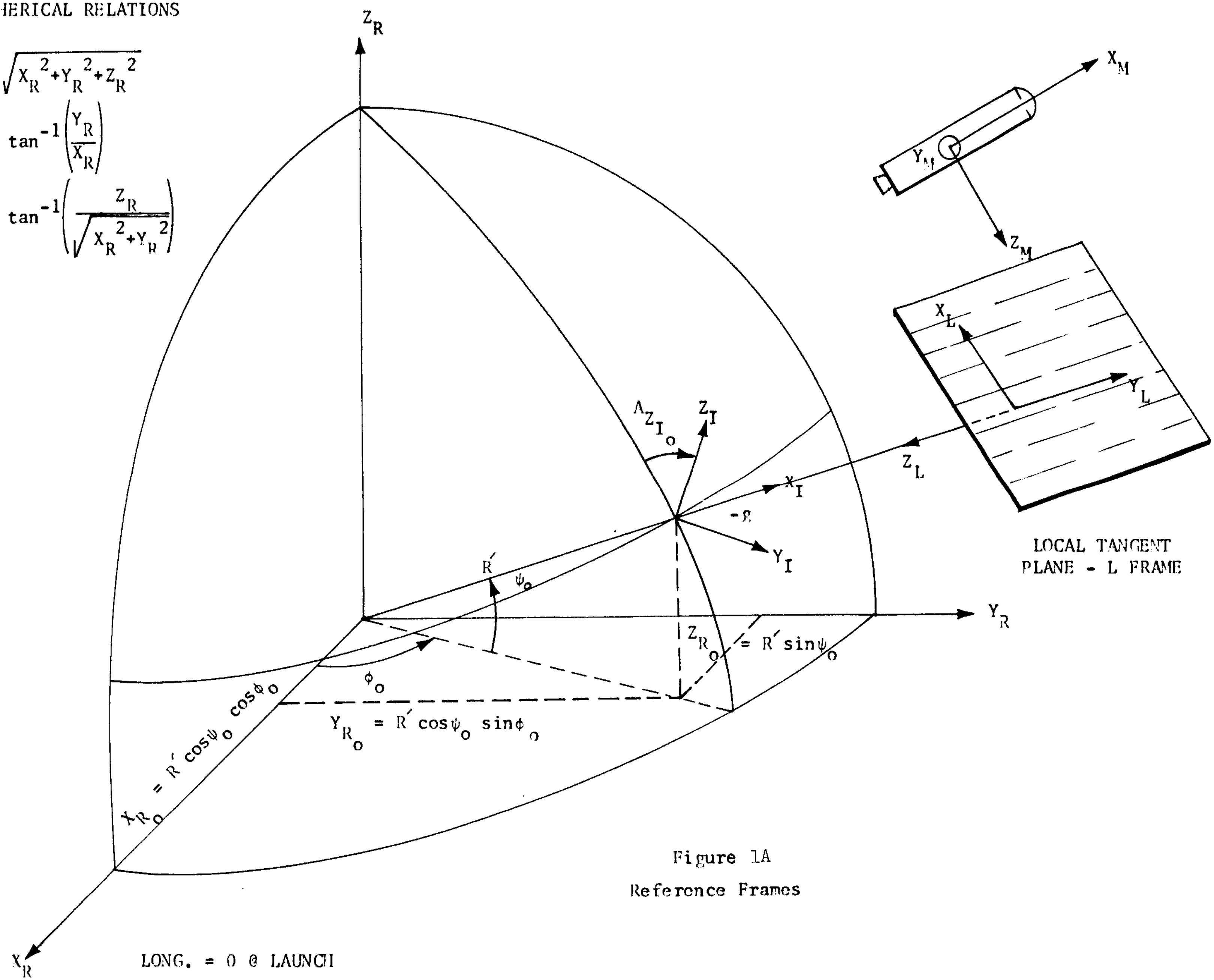

Figure 1A
Reference Frames

ORBITAL TRANSFER BY OPTIMUM THRUST DIRECTION AND DURATION

by

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ABSTRACT

A three dimensional derivation is presented of the equations and boundary conditions necessary to determine the minimum fuel orbit transfer path by optimizing the thrust direction and duration. The formulation, known as the Mayer problem in the calculus of variations, yields a two point boundary value problem. A Newton-Raphson method was used to attempt convergence of this two point boundary value problem, but it was found to be inadequate. However, with the final orbit unspecified numerous solutions satisfying the Mayer formulation were generated and then compared with the optimum two-impulse transfer between the same two orbits. This comparison is quite revealing; it shows first, that for the restricted class of orbits examined the optimum two-impulse estimate of velocity increment, or fuel required is very good. Second, it demonstrates that although the optimum departure and arrival points obtained from the impulsive and finite thrust solutions may be quite different, the penalty in using the former for design estimates may be quite minor.

INTRODUCTION

In this report we are concerned with the problem of moving a vehicle between two arbitrary orbits in space. The orbits are assumed to have one planet as a common focus which generates a uniform central gravitational field, and the vehicle is assumed to be capable of thrust direction and on-off control. We present a complete derivation, in three dimensions, of the equations and boundary conditions necessary to determine the minimum-fuel orbit transfer path by optimizing the thrust direction and duration, and the departure and arrival points on the initial and final orbits. The Mayer formulation of the calculus of variations is used.

We turn to optimization procedures for finding the transfer path for three reasons: First, the problem of realistic minimum fuel requirements for space maneuvers is one of extreme importance. Second, for the purposes of design studies based on impulsive transfer, it is necessary to know the error made by the assumption of impulses. Third, the optimization technique gives an organized and general way for finding a transfer path; it is a procedure that is of significance no matter what quantity is to be extremized, since it provides a suitable steering program to accomplish the desired mission.

Selection of the optimization technique is primarily decided by what has been reported in the literature, and the experience of the investigator. Either the indirect method-use of Lagrange multipliers-or the direct method-steepest descent-can be used. Reference (2) reports a successful application of the Mayer formulation to the problem of boosting the maximum payload into orbit with a high thrust engine. Reference (3) also uses the same method successfully on the problem of coplanar orbital transfer with very low thrust engines. Both applications utilized the Newton-Raphson method as the principal iterative technique for solving the two-point boundary value problem. These reports were the main factors in this selection and in the initial approach to the two-point boundary value problem used in this study.

I. EQUATIONS OF MOTION*

The kinetic energy per unit mass is:

$$P = 1/2\ (\dot{r}^2 + r^2\dot{\theta}^2 + r^2\cos^2\theta\ \dot{\phi}^2);\ \text{see Fig. 1.}$$

The potential energy per unit mass is:

$$V = \frac{-\mu}{r}\quad (\mu = K\ M_{earth})$$

The Lagrangian, L = P - V:

$$L = 1/2\ (\dot{r}^2 + r^2\dot{\theta}^2 + r^2\cos^2\theta\,\dot{\phi}^2) + \frac{\mu}{r}$$

The three second-order equations of motion are obtained from:

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = Q_i,\quad i = 1\text{-}3,$$

where the q_i are r, θ, and ϕ. The Q_i are the generalized force and moments due to the thrust, T:

$$Q_r = \frac{T}{m}\cos\psi\cos\nu$$

$$Q_\theta = \frac{T}{m}\ r\sin\psi$$

$$Q_\phi = \frac{T}{m}\ r\cos\psi\sin\nu\cos\theta$$

Thus, the three second-order equations of motion are:

$$\ddot{r} - r\,\dot{\theta}^2 - r\cos^2\theta\,\dot{\phi}^2 + \frac{\mu}{r^2} = \frac{T}{m}\cos\psi\cos\nu \tag{1}$$

$$\frac{d}{dt}(r^2\,\dot{\theta}) + r^2\cos\theta\sin\theta\,\dot{\phi}^2 = \frac{T}{m}\ r\sin\psi \tag{2}$$

*See also references 4-6.

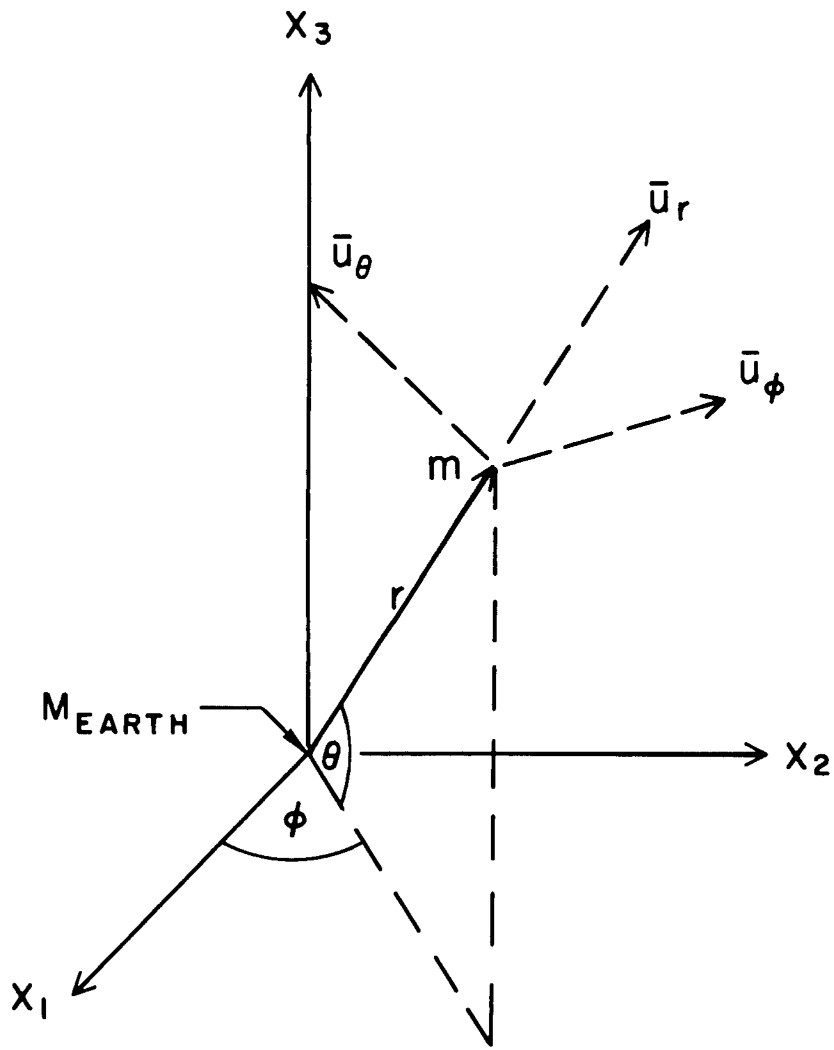


FIG. 1a.

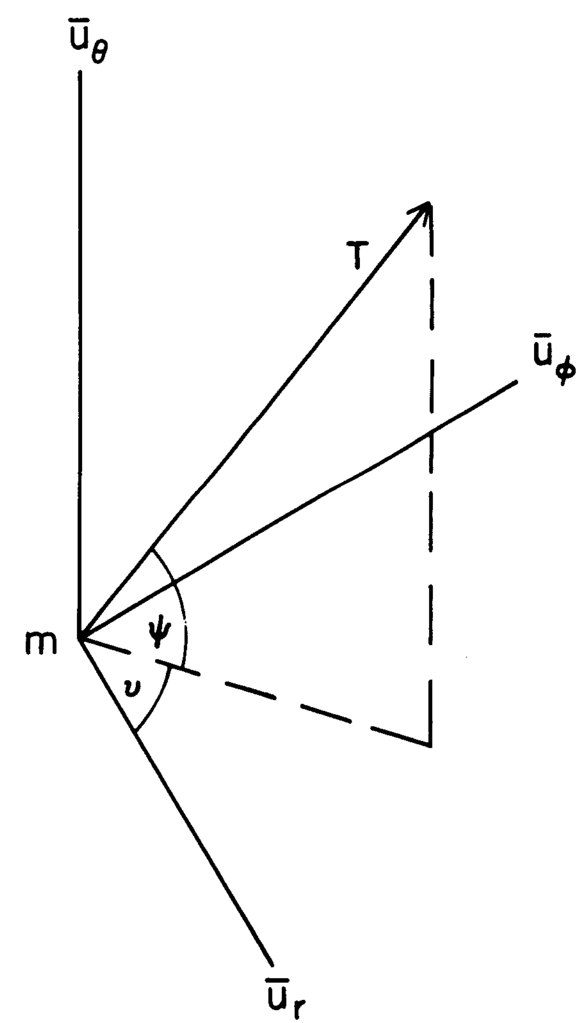


FIG. 1b.

COORDINATE SYSTEMS

$$\frac{d}{dt}(r^2 \cos^2 \theta \dot{\phi}) = \frac{T}{m} r \cos\psi \sin\nu \cos\theta \tag{3}$$

We want the thrust, T, to be either on or off. Hence, we define $\dot{T} = c\beta$, where c = an effective exhaust velocity, and β = mass flow rate.

$$\text{Check dimensions:} \quad \left[T\right] = F = \frac{ML}{T^2}; \quad \left[c\beta\right] = \frac{L}{T} \cdot \frac{M}{T}$$

Expanding (2) and (3), and noting that we cannot have $\theta = \pm\frac{\pi}{2}$, we get the following seven first-order equations of motion, where new variables ρ, x, y are defined as indicated:

$$w_1 \equiv \dot{r} - \rho = 0 \tag{4}$$

$$w_2 \equiv \dot{\theta} - x = 0 \tag{5}$$

$$w_3 \equiv \dot{\phi} - y = 0 \tag{6}$$

$$w_4 \equiv \dot{\rho} - r x^2 - r\cos^2\theta\, y^2 + \frac{\mu}{r^2} - \frac{c\beta}{m}\cos\psi\cos\nu = 0 \tag{7}$$

$$w_5 \equiv \dot{x} + \frac{2\rho x}{r} + \cos\theta\sin\theta\, y^2 - \frac{c\beta}{mr}\sin\psi = 0 \tag{8}$$

$$w_6 \equiv \dot{y} - 2\tan\theta \cdot x y + \frac{2\rho y}{r} - \frac{c\beta\cos\psi\sin\nu}{mr\cos\theta} = 0 \tag{9}$$

$$w_7 \equiv \dot{m} + \beta = 0 \tag{10}$$

The optimum path (for min. fuel expenditure) that is to be found must satisfy the equations of motion, and this is represented by constraints, $w_i = o$, $i = 1\text{-}7$.

There is one further constraint to be added: We require the thrust to be on or off--no throttling. This is expressed by:

$$w_8 \equiv \beta(\beta - \beta_{max.}) = 0$$

Hence, problem variables are:

Dependent			Independent
Dynamic and kinematic		Control	
r	ρ	ψ	t
θ	x	ν	
ϕ	y	β	
m			

Denoting all dependent variables by z, the constraints can be expressed as:

$$w_i = \dot{z}_i - f_i(z_j) = 0 \quad i = 1 - 8, \; j = 1 - 10$$

II. DERIVATION OF OPTIMIZATION PROBLEM*

A. Since the quantity that we want to minimize only enters in the boundary conditions (we use the Mayer formulation of the calculus of variations), let us first obtain the Euler-Lagrange equations associated with the control variables ψ, ν, β.

$$F = \lambda_i(t)\, w_i(\dot{z}_i, z_j)$$

Require:

$$\frac{d}{dt}\frac{\partial F}{\partial \dot{z}_i} - \frac{\partial F}{\partial z_i} = 0 \qquad z_i = \nu, \psi, \beta$$

$$\frac{\partial F}{\partial \dot{z}_i} = 0 \quad \therefore \quad \frac{\partial \lambda_j w_j}{\partial z_i} = \lambda_j \frac{\partial w_j}{\partial z_i} = 0$$

1.) $z = \nu$

$$\lambda_4\left(\frac{c\beta}{m}\cos\psi\sin\nu\right) + \lambda_6\left(\frac{-c\beta\cos\psi\cos\nu}{mr\cos\theta}\right) = 0$$

$$\frac{c\beta}{m}\left[\lambda_4\cos\psi\sin\nu - \lambda_6\frac{\cos\psi\cos\nu}{r\cos\theta}\right] = 0$$

If $\beta = 0$, then $T = 0$ and ψ and ν have no meaning, and we simply compute the $\lambda_i(t)$ by a closed-form solution which is given in Appendix A. For $\beta \neq 0$, and c and $m \neq 0$ for all t:

$$\cos\psi\left[\lambda_4\sin\nu - \frac{\lambda_6\cos\nu}{r\cos\theta}\right] = 0$$

$$\therefore \text{Either } \psi = \pm\frac{\pi}{2}, \text{ or } \tan\nu = \frac{\lambda_6}{\lambda_4 r\cos\theta} \qquad (11)$$

$$\therefore \sin\nu = \frac{\lambda_6}{\pm D_\nu}, \quad \cos\nu = \frac{\lambda_4 r\cos\theta}{\pm D_\nu} \qquad (12), (13)$$

---

*See also references (7)-(9).

where

$$D_\nu = \sqrt{\lambda_6^2 + \lambda_4^2 r^2 \cos^2\theta}$$

Note: From equations (7) and (9), that if $\psi = \pm\pi/2$, the ν terms drop out, as expected on physical grounds.

2.) $z = \psi$

$$\lambda_4 \left[\frac{c\beta}{m} \sin\psi \cos\nu\right] + \lambda_5 \left[\frac{-c\beta}{mr} \cos\psi\right] + \lambda_6 \left[\frac{c\beta \sin\psi \sin\nu}{mr \cos\theta}\right] = 0$$

If $\beta = 0$, then the argument is the same as above. For $\beta \neq 0$:

$$\sin\psi \left[r\lambda_4 \cos\nu + \frac{\lambda_6 \sin\nu}{\cos\theta}\right] - \lambda_5 \cos\psi = 0$$

Insert (12) and (13) for $\sin\nu$ and $\cos\nu$, and collect terms:

$$\sin\psi \left[\frac{\pm D_\nu}{\cos\theta}\right] - \lambda_5 \cos\psi = 0$$

$$\frac{\sin\psi}{\cos\psi} = \tan\psi = \frac{\lambda_5 \cos\theta}{\pm D_\nu} \qquad (14)$$

3.) $z = \beta$

$$\lambda_4 \left[\frac{-c}{m} \cos\psi \cos\nu\right] + \lambda_5 \left[\frac{-c}{mr} \sin\psi\right] + \lambda_6 \left[\frac{-c\cos\psi \sin\nu}{mr\cos\theta}\right]$$

$$+ \lambda_7 (1) + \lambda_8 \left[(\beta - \beta_{max}) + \beta\right] = 0 \qquad (15)$$

This equation yields λ_8, but it is of no significance in this problem.

B. To reconcile the sign ambiguities in 1.) and 2.), above, and to determine when $\beta = 0$, $\beta = \beta_{max.}$, we turn to the Weierstrass necessary condition.

1.) This condition states that for a minimum, $E \geqslant 0$:

$$E = F(Z_i^*, \dot{Z}_i^*) - F(Z_i, \dot{Z}_i) - \sum_i (\dot{Z}_i^* - \dot{Z}_i) \frac{\partial F}{\partial \dot{Z}_i}$$

Z_i^* differs from Z_i by a <u>finite</u>, but admissible amount.

The only variables which admit of such strong variation are ν, ψ, and β, where, for example:

$$\psi = \psi \text{ or } \psi + \pi; \quad \nu = \nu \text{ or } \nu + \pi; \quad \beta = 0 \text{ or } \beta_{max}.$$

Now, the third term in E is identically zero since there are no constraints involving $\dot{\psi}$, $\dot{\nu}$, $\dot{\beta}$.

$$E = \lambda_i(t)\, w_i\,(Z_j^*, \dot{Z}_i^*) - \lambda_i(t)\, w_i\,(Z_j, \dot{Z}_i) \geq 0 \qquad j, i = 1\text{-}8$$

$$E = \lambda_i(t) \left[\dot{Z}_i^* - f_i\,(Z_j^*)\right] - \lambda_i(t) \left[\dot{Z}_i - f_i\,(Z_j)\right] \geqslant 0$$

$$E = \lambda_i(t)\, f_i\,(Z_j) - \lambda_i(t)\, f_i\,(Z_j^*) \geq 0$$

or

$$\lambda_i(t)\, f_i\,(Z_j) \geq \lambda_i(t)\, f_i\,(Z_j^*) \tag{16}$$

Applying (16) we get:

$$\lambda_4 \left[\frac{c\beta}{m} \cos\psi \cos\nu\right] + \lambda_5 \left[\frac{c\beta}{mr} \sin\psi\right] + \lambda_6 \left[\frac{c\beta \cos\psi \sin\nu}{mr \cos\theta}\right]$$

$$+ \lambda_7 (-\beta) + \lambda_8 \left(-\beta\left[\beta - \beta_{max}\right]\right) \geqslant \lambda_4 \left[\frac{c\beta^*}{m} \cos\psi^* \cos\nu^*\right]$$

$$+ \lambda_5 \left[\frac{c\beta^*}{mr} \sin\psi^*\right] + \lambda_6 \left[\frac{c\beta^* \cos\psi^* \sin\nu^*}{mr \cos\theta}\right] + \lambda_7 (-\beta^*)$$
$$+ \lambda_8 \left[-\beta^* (\beta^* - \beta_{max})\right]$$

Note, first, that the λ_8 term $\equiv 0$.

Now, factoring out a β and β^* yields, in the notation of ref. (8):

$$\beta k - \beta^* k^* \geqslant 0$$

where

$$k = \frac{c}{m}\left(\lambda_4 \cos\psi \cos\nu + \frac{\lambda_5}{r}\sin\psi + \frac{\lambda_6 \cos\psi \sin\nu}{r\cos\theta}\right) - \lambda_7$$

For $k = k^*$, $\beta \neq \beta^*$; $k(\beta - \beta^*) \geqslant 0$

If $k > 0$, then $\beta > \beta^* \Rightarrow \beta = \beta_{max}$ (17a)

If $k < 0$, then $\beta < \beta^* \Rightarrow \beta = 0$ (17b)

Thus, we have the engine on-off criteria.

For $\beta = \beta^*$, $k \neq k^*$;

$$\lambda_4 \cos\psi \cos\nu + \frac{\lambda_5}{r}\sin\psi + \frac{\lambda_6 \cos\psi \sin\nu}{r\cos\theta} \geqslant \lambda_4 \cos\psi^* \cos\nu^*$$

$$+ \frac{\lambda_5}{r}\sin\psi^* + \frac{\lambda_6 \cos\psi^*}{r\cos\theta}\sin\nu^* \tag{18}$$

a.) $\psi = \psi^*$; $\nu \neq \nu^* \Longrightarrow \nu = \nu$ or $\nu + \pi\ (= \nu^*)$

Hence, (18) becomes

$$\lambda_4 \cos\psi \cos\nu + \frac{\lambda_6 \cos\psi \sin\nu}{r\cos\theta} \geqslant 0$$

Using (12) and (13):

$$\cos\psi \, \frac{\lambda_4^2 r^2 \cos^2\theta + \lambda_6^2}{\pm D_\nu \, r\cos\theta} \geq 0$$

or,

$$\cos\psi \left[\frac{\pm D_{\nu}}{r \cos\theta} \right] \geq 0$$

Since $r > 0$, and $\frac{-\pi}{2} < \theta < \frac{\pi}{2}$, the above yields

$$\left. \begin{array}{l} + D_{\nu}, \ \cos\psi > 0 \\ \\ - D_{\nu}, \ \cos\psi < 0 \end{array} \right] \tag{19}$$

Physically, we will most probably be confined to

$$\frac{-\pi}{2} < \psi < \frac{\pi}{2} \Rightarrow + D_{\nu} .$$

b.) $\nu = \nu^{*}$; $\psi \neq \psi^{*} \Rightarrow \psi = \psi$ or $\psi + \pi \ (= \psi^{*})$

From (14) and (19):

$$\tan\psi = \frac{\lambda_5 \cos\theta}{\pm D_{\nu}}$$

$$\sin\psi = \frac{\lambda_5 \cos\theta}{\pm D_{\psi}}, \quad \cos\psi = \frac{\pm D_{\nu}}{\pm D_{\psi}} \tag{20}$$

where

$$D_{\psi} = \sqrt{D_{\nu}^2 + \lambda_5^2 \cos^2\theta}$$

From (18) again:

$$\lambda_4 \cos\psi \cos\nu + \frac{\lambda_5 \sin\psi}{r} + \frac{\lambda_6 \cos\psi \sin\nu}{r \cos\theta} \geq 0$$

Substituting (12), (13), and (20) and clearing yields:

$$\frac{\pm D_{\psi}}{r \cos \theta} \geq 0$$

Again, since $r \cos \theta > 0$, this requires $+ D_{\psi}$ (21)

C. There is a first integral, since the Lagrangian, F, does not involve time explicitly.

$$\frac{\partial F}{\partial \dot{z}_k} \dot{z}_k = C; \quad \frac{\partial \lambda_i(t) w_i (\dot{z}_i, z_j)}{\partial \dot{z}_k} \dot{z}_k = C$$

$$\lambda_i(t) \frac{\partial (\dot{z}_i - f_i (z_j))}{\partial \dot{z}_k} \dot{z}_k = C$$

Hence,

$$\lambda_1 \dot{r} + \lambda_2 \dot{\theta} + \lambda_3 \dot{\phi} + \lambda_4 \dot{\rho} + \lambda_5 \dot{x} + \lambda_6 \dot{y} + \lambda_7 \dot{m} = C \tag{22}$$

D. Boundary Conditions

The boundary conditions to be applied come from two sources: Those implied by the physics of the problem, and the remainder from the transversality condition

$$\left[dG + (F - \frac{\partial F}{\partial \dot{z}_k} \dot{z}_k) \, dt + \frac{\partial F}{\partial \dot{z}_k} dz_k \right]_0^T = 0, \tag{23}$$

where G is the function to be minimized.

1.) To clarify the derivation of the boundary conditions, let us first consider that the two orbits are coplanar. We reiterate the problem: Find the minimum fuel path to transfer between two coplanar orbits by optimizing the thrust direction (ν) and duration ("Bang-bang" control). The departure and arrival points on the initial and final orbits are not specified, but the total time of transfer is specified. The geometry is shown in Figure 2.

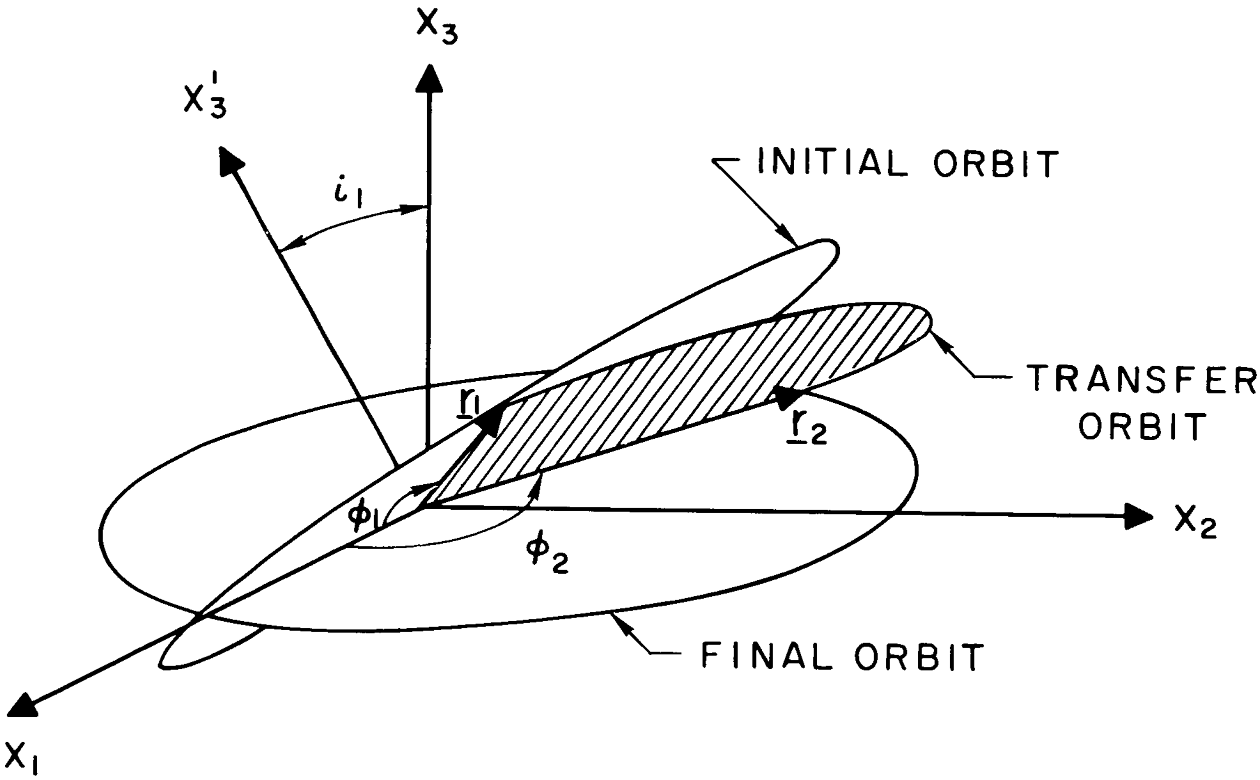


FIG. 2. TRANSFER GEOMETRY

Thus, we have a system of 10 first-order differential equations for the variables:

$$r,\ \phi,\ \rho,\ y,\ m,\ \lambda_1,\ \lambda_3,\ \lambda_4,\ \lambda_6,\ \lambda_7$$

This system thus requires 10 boundary conditions. The seven specified by the physics of the problem are: (i = initial, f = final).

$$p_i \text{ (or } h_i\text{)},\ e_i \text{ (or } E_i\text{)},\ \omega_i,\ m_i$$

$$p_f \text{ (or } h_f\text{)},\ e_f \text{ (or } E_f\text{)},\ \omega_f \tag{24}$$

p, e, ω are semi-latus rectum, eccentricity, and argument of perigee, respectively. h and E are angular momentum and total energy.

We derive the three remaining conditions from Equation (23) and thus we are obliged finally to select the quantity to be optimized. Since we wish to compare our results with minimum impulsive orbital transfer, let us consider minimizing the characteristic velocity,

$$G = c \ln \frac{m_i}{m_f}$$

Equation (23) becomes, utilizing (22);

$$\left[\frac{-c}{m_f} + \lambda_7\right] dm_f + \left[\frac{c}{m_i} - \lambda_7\right] dm_i + \Big[- C\,dt + \lambda_1\,dr$$

$$+ \lambda_3\,d\phi + \lambda_4\,d\rho + \lambda_6\,dy\Big]_0^T = 0 \tag{25}$$

Since m_i is specified, $dm_i = 0$. Also, $dt\Big]_0^T = 0$, which implies C = unknown. Thus,

$$\lambda_7 = \frac{c}{m} \text{ at } t = T \tag{26}$$

This is our eighth boundary condition. The remaining two come from

$$\left[\lambda_1\,dr + \lambda_3\,d\phi + \lambda_4\,d\rho + \lambda_6\,dy\right]_0^T = 0, \tag{27}$$

where we use orbit equations to relate the differentials in terms of the given parameters p, e, and w. To do this we note:

$$r = \frac{p}{1 + e\cos(\phi - \omega)} \equiv f(\phi) \tag{28}$$

$$dr = f'(\phi)\, d\phi$$

$$E/m = 1/2(\dot{r}^2 + r^2\dot{\phi}^2) - \frac{\mu}{r} \tag{29}$$

$$d(E/m) = \dot{r}d\dot{r} + \dot{\phi}^2\, r dr + r^2\dot{\phi}\, d\dot{\phi} + \frac{\mu}{r^2} dr = 0$$

$$h/m = r^2\dot{\phi} \tag{30}$$

$$d(h/m) = 2r\dot{\phi} dr + r^2 d\dot{\phi} = 0$$

Expressing all the differentials in terms of $d\phi$, the two boundary conditions then are

$$f'(\phi)\left[\lambda_1 - \frac{\lambda_4}{\dot{r}}\left(- r\dot{\phi}^2 + \frac{\mu}{r^2}\right) - \frac{2\lambda_6\dot{\phi}}{r}\right] + \lambda_3 = 0, \text{ at } t = 0,\ T.$$

These two equations can be put in a more revealing form. Substituting $\dot{\rho}$ and $\dot{y}$ from the equations of motion, we find

$$\dot{r}\lambda_1 + \dot{\phi}\lambda_3 + \dot{\rho}\lambda_4 + \dot{y}\lambda_6 = \frac{c\beta}{m}\left[\frac{\lambda_6 \sin\nu}{r} + \lambda_4\cos\nu\right] \text{ at } t = 0,\ T$$

Utilizing Equation 17 from p. 9, with $\psi \equiv 0$ and $\lambda_5 \equiv 0$, we see that the right side of the above equation is

$$\beta k + \beta\lambda_7, \text{ or}$$

$$\dot{r}\lambda_1 + \dot{\phi}\lambda_3 + \dot{\rho}\lambda_4 + \dot{y}\lambda_6 + \dot{m}\lambda_7 = \beta k \text{ at } t = 0,\ T \tag{31}$$

This thus identifies the constant, C (Equation (22)) as equal to βk at the end points.

Further, if $C \neq 0$ at $t = 0$, (31) implies that $k(0) = k(T)$.

2.) We can now proceed to derive, rather succintly, the boundary conditions for the three dimensional case. The problem requires fourteen boundary conditions since there are fourteen first order differential conditions for the variables:

$$r, \theta, \phi, \rho, x, y, m, \lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6, \lambda_7.$$

The physics of the problem now yields eleven conditions while the transversality (Equation (23)) yields three, exactly as in the planar case. The additional four physical constraints are that the vehicle's position and velocity are to be in the specified initial and final planes.

We list the fourteen conditions in terms of their origin:

(a) From the final point ($t = T$), there are five: By choosing the final plane to have zero inclination the two additional constraints at the final point are simply $\theta(T) = 0$ and $\dot{\theta}(T) = 0$. The other three are Equations (28), (29) and (30) applied to the final point.

(b) From the initial point ($t = 0$), there are six: One of the six is the specification of initial mass, while five are orbit equations. The initial orbital plane is taken to have an inclination i and to have its ascending node on the x_1 axis as in Figure 2. The departure point angle called ϕ_1 in Figure 2 is replaced so that ϕ represents the angle in the x_1, x_2 plane as in Figure 1. The five orbital equations may be taken as: Equations (28), (29), (30),

$$\sin \phi = \tan \theta \cot i, \tag{33}$$

and

$$yr^2 \cos^2\theta = \frac{h}{m} \cos i. \tag{34}$$

(c) From the transversality condition, there are three:

$$\lambda_7 = \frac{c}{m} \text{ at } t = T \tag{35}$$

is obtained exactly as before. The remaining two equations are:

$$\left[\lambda_1 \, dr + \lambda_2 \, d\theta + \lambda_3 \, d\phi + \lambda_4 \, d\rho + \lambda_5 \, dx + \lambda_6 \, dy \right]_{t = 0} = 0 \tag{36}$$

and

$$\left[\lambda_1 \, dr + \lambda_3 \, d\phi + \lambda_4 \, d\rho + \lambda_6 \, dy \right]^{t = T} = 0 \tag{37}$$

In addition it should be pointed out that just as in the planar case Equation (36) and Equation (37) are equivalent to

$$\beta(0)\ k\ (0) = C \tag{38}$$

$$\beta(T)\ k\ (T) = C \tag{39}$$

Finally, for use in computation it must be indicated that equation (36) along with the total differentials of the five orbit equations (28, 29, 30, 33, and 34) constitute a set of six homogenous equations, the determinant of whose coefficients is the required relationship. This is the generalization of Equation (31) for the initial point. For the final point the generalization is the same as in the planar problem.

E. Corner Conditions

The points at which the thrust goes on or off give rise to discontinuities in the $\dot{z}_k$. The mathematical criterion needed to join different positions of the extremal arc is supplied by the Erdmann-Weierstrass corner condition:

$$\left(\frac{\partial F}{\partial \dot{z}_k}\right)_- = \left(\frac{\partial F}{\partial \dot{z}_k}\right)_+$$

or

$$\left(\lambda_k\right)_- = \left(\lambda_k\right)_+ , \quad k = 1-7 \tag{40}$$

$$\left[- F + \frac{\partial F}{\partial \dot{z}_k}\ \dot{z}_k\right]_- = \left[- F + \frac{\partial F}{\partial \dot{z}_k}\ \dot{z}_k\right]_+$$

or,

$$C_- = C_+ \tag{41}$$

We observe that any of the seven conditions which comprise (40) would not apply if the value of the physical variable were specified at the discontinuity. Similarly, (41) would not apply if the time of the discontinuity were specified.

F. Euler-Lagrange Equations

Here we write down the differential equations for the Lagrange multipliers, which come from the Euler necessary condition in the calculus of variations:

$$\frac{d}{dt}\left[\frac{\partial F}{\partial \dot{z}_k}\right] - \frac{\partial F}{\partial z_k} = 0; \quad z_k = r, \theta, \phi, \rho, x, y, m \tag{42}$$

$$F = \lambda_j w_j = \lambda_j (t) \left[\dot{z}_j - f_j (z_\ell)\right]$$

$$\frac{d}{dt}\left(\lambda_j(t)\, \delta_{jk}\right) = - \lambda_j \frac{\partial f_j}{\partial z_k} (z_\ell)$$

$$\dot{\lambda}_k = - \lambda_j \frac{\partial f_j}{\partial z_k} (z_\ell) \tag{43}$$

Using equations (4) - (10), equation (43) yields:

$$\dot{\lambda}_1 = - \lambda_4 \left[x^2 + y^2 \cos^2 \theta + \frac{2\mu}{r^3}\right] - \frac{\lambda_5}{r^2} \left[2 \rho x - \frac{c\beta \sin \psi}{m}\right]$$

$$- \frac{\lambda_6}{r^2} \left[2 \rho y - \frac{c \beta \cos \psi \sin \nu}{m \cos \theta}\right] \tag{44}$$

$$\dot{\lambda}_2 = - \lambda_4 (- 2ry^2 \cos \theta \sin \theta) + \lambda_5 y^2 \cos 2 \theta$$

$$- \lambda_6 \left[2 x y \sec^2 \theta + \frac{c \beta}{mr} \cos \psi \sin \nu \tan \theta \sec \theta\right] \tag{45}$$

$$\dot{\lambda}_3 = 0 \tag{46}$$

$$\dot{\lambda}_4 = - \lambda_1 + \frac{2 \lambda_5 x}{r} + \frac{2 \lambda_6 y}{r} \tag{47}$$

$$\dot{\lambda}_5 = - \lambda_2 - 2r x \lambda_4 + \frac{2 \lambda_5 \rho}{r} - 2 \lambda_6 y \tan \theta \tag{48}$$

$$\dot{\lambda}_6 = - \lambda_3 - 2ry \lambda_4 \cos^2 \theta + 2 \lambda_5 y \cos \theta \sin \theta - 2 \lambda_6 \left[x \tan \theta - \frac{\rho}{r}\right] \tag{49}$$

$$\dot{\lambda}_7 = \frac{\beta}{m} \left[\frac{c}{m} \lambda_4 \cos \psi \cos \nu + \frac{c \lambda_5 \sin \psi}{mr} + \frac{c \lambda_6 \cos \psi \sin \nu}{mr \cos \theta}\right. \tag{50}$$

or,

$$\dot{\lambda}_7 = \frac{\beta}{m} \left[k + \lambda_7\right] ,$$

from Section II-B-1.

III. ITERATIVE METHOD

The equations (4) - (10) and (44) - (50), plus the control equations for the switching function, k, and the steering angles, ψ and ν, are a set of differential and algebraic equations whose boundary values at t = 0 and t = T must meet the specified conditions at those two points. We are thus faced with the well-known two-point boundary value problem. The Newton-Raphson method, and a "Matrix Modification" technique were selected as the first iterative techniques to attempt convergence of the two-point boundary value problem. Both these methods are fully explained in reference (2), and only a brief description of the convergence characteristics of this method on this problem will be given here.

The iterative techniques have so far been only applied to the coplanar case because it was felt that until a fast and reliable method was available for that problem it was rather hopeless to tackle the three-dimensional case. Reference (3) reported success with this technique for low-thrust engines, but in this case when the thrust-to-weight ratio (T/W) is between one and ten, it does not seem to be able to handle the problem. One comment about a T/W of ten is in order; the iterative procedure begins by first obtaining the optimum two-impulse transfer. We then have the optimum departure and arrival points, velocity increment necessary, time for the transfer, and initial and final thrust direction. Hence, if we assume an engine with a T/W = 10, we have almost an impulsive vehicle, and if the final time is set equal to the impulsive time for transfer plus the time necessary to burn fuel yielding a velocity increment equal to or slightly greater than the impulsive solution, we can expect that the finite-thrust solution will be very close to the impulsive solution in all respects. Once this one has been obtained, we can then proceed to decrease the T/W to 8, 6, 4, etc., obtaining solutions for all these, until we are down to precisely the engine in which we are interested.

Now, the Newton-Raphson method applied to the coplanar problem has the behavior of converging on the transversality condition first, equation (31), and then keeping that satisfied, move very slowly towards meeting the orbit conditions, p, e, and ω. The conclusion, so far, is that the method is inadequate for this complex and sensitive problem. However, several modifications of the method, and its use, are being studied, and it may yet prove capable. If not, other iterative methods for handling the two-point boundary value problem are being studied, and will be tried if the Newton-Raphson proves conclusively unsatisfactory.

NUMERICAL RESULTS

In the introduction to this paper, three reasons for turning to optimization procedures for the solution of the minimum fuel orbital transfer problem were given. This section gives an indication of the answer to the second statement; i.e., the comparison with two-impulse orbital transfer. The answer is not conclusive since the switching function time history was restricted to one coast period, and the second burn period was terminated as soon as

$$k(t) = k(0) \text{ ; see equation (31).}$$

Thus, a rather restricted class of initial and final orbits was considered; all orbit pairs intersected, and in most cases the intersection was quite shallow.

The following table presents some of the results gathered from this restricted comparison. The first column is the thrust-to-weight ratio at the initial orbit; for example, a vehicle of 1000 slugs mass, with fuel-flow rate, β, of 1 slug/sec., has a specific impulse of 300 sec. if the $(T/W)_i = .7118$, at a distance of 6058 miles from the center of the earth. In the second column, the percentage difference in velocity increment is given; $V_F = c \ln \frac{m_i}{m_f}$, and V_I is equal to the total velocity increment from the two-impulse minimization. Total $\Delta\phi$, in the third column was computed as follows:

$$\text{Total } \Delta\phi = \left|\phi_{i,F} - \phi_{i,I}\right| + \left|\phi_{f,F} - \phi_{f,I}\right|$$

Thus it represents the total deviation in the departure and arrival points between this finite thrust solution—subscript F—and the impulsive solution—subscript I. The last column gives an approximation to the penalty in velocity increment, or fuel, if the departure and arrival point of the impulsive solution is used instead of the points specified by the finite thrust solution. This estimate was obtained in the following way: Reference (1) presents contour maps of minimum transfer velocity on a ϕ_i, ϕ_f plot. By differencing the value at $(\phi_{i,I}, \phi_{f,I})$ with the value at $(\phi_{i,F}, \phi_{f,F})$, and dividing by V_I, we obtain an estimate of the penalty in velocity, or fuel, that would be incurred. We emphasize that this is an approximation; but in view of the results in the second column, it is probably a reasonable one.

Finite Thrust Versus Two Impulse Comparison

$(T/W)_i$	$\frac{(V_F - V_I)}{V_I} 10^2$	Total $\Delta\phi$, deg.	$\frac{\Delta V}{V_I} 10^2$ Penalty
10	.086	20.0	.135
10	.136	26.3	.410
8	.203	21.1	.352
8	.236	27.9	.401
6	.143	18.7	.365
6	.501	29.8	.685
4	.278	34.4	.874
4	.354	32.2	.247
2	.224	24.8	.611
2	.293	28.7	.631
.7118	.095	72.8	1.89
.7118	.194	13.0	.407

We observe from the first and second columns, that if orbit transfers with realistic vehicles are restricted to be completed in one orbit, then the time constraint—obtained from the impulsive solution—placed upon these finite thrust solutions is also realistic, and, ipso facto, the fuel requirement for the transfer obtained from the two-impulse solution is a very good estimate of that which would actually be needed. This is, of course, with the assumption that the finite thrust transfer vehicle departs and arrives at the proper point, for we see that the discrepancies in ϕ_i and ϕ_f can be quite sizable. However, from the fourth column, we note that the penalty in fuel, or velocity, for using the optimum ϕ_i, ϕ_f from the impulsive solution rather than those specified by the finite thrust solution may be quite minor; however, this was a rather restricted comparison, and a good deal more numerical results are necessary before any even tentative generalizations in this direction are possible.

CONCLUDING REMARKS

The Mayer formulation of the calculus of variations has been used to derive, in three dimensions, the equations and boundary conditions necessary to determine the minimum fuel orbit transfer path by optimizing the thrust direction and duration, and the departure and arrival points on the initial and final orbits. The closed-form solution to the Euler-Lagrange equations, which apply along the coast arc has also been derived, rather explicitly, and has been verified by some of the numerical integrations indicated in the preceding section.

The numerical results section is considerably leaner than desired. One conclusion, therefore, is that the multivariable Newton-Raphson iteration technique is inadequate for this complex and sensitive problem. This is a useful, albeit frustrating result. A more gratifying result is the favorable comparison of two-impulse and finite thrust orbit transfer solutions. Restrictive as it is, it should be of interest to design personnel, for it is the first proven indication, to this writer's knowledge, of the real utility of the impulsive solution and how much a design based on it differs from the optimum.

It is hoped, and rather optimistically felt, that one of the iteration techniques currently under study for solving the two-point boundary value problem will be effective in this endeavor. With this accomplished, an unrestricted variety of problems with an equally unrestricted genus of propulsion systems will be able to be expediently solved. The two-impulse solution is obviously not universally a good estimate for design, or even applicable. When low-thrust ion or nuclear propulsion systems are being considered, and interplanetary transfers are being studied, it will be distinctly advantageous, if not imperative, that the capability begun herein be a reality.

APPENDIX A

SOLUTION TO EULER-LAGRANGE EQUATIONS DURING COAST

With the thrust off ($\beta = 0$), the equations of motion are

$$\ddot{r} = r\dot{\phi}^2 - \frac{\mu}{r^2} \tag{A1}$$

$$r\ddot{\phi} = -2\,\dot{r}\dot{\phi} \tag{A2}$$

for coplanar orbits. The solution to these involves four arbitrary constants; p_C, e_C, ω_C - the elements of the coast orbit - and ϕ_C, the angle at which the coast is begun.

The Euler-Lagrange equations are:

$$\dot{\lambda}_4 = -\lambda_1 + \frac{2\lambda_6\dot{\phi}}{r} \tag{A3}$$

$$\dot{\lambda}_6 = -\lambda_3 - 2\,\lambda_4\, r\dot{\phi} + \frac{2\lambda_6\dot{r}}{r} \tag{A4}$$

$$\lambda_1 = \frac{1}{\dot{r}}\left[C - \lambda_3\dot{\phi} - \lambda_4\ddot{r} - \lambda_6\ddot{\phi}\right] \tag{A5}$$

$$\dot{\lambda}_7 = 0;\ \lambda_7 = \lambda_7 \text{ at beginning of coast} \tag{A6}$$

First, change the independent variable from t to ϕ:

$$\lambda'_4\,\dot{\phi} = -\lambda_1 + \frac{2\lambda_6\dot{\phi}}{r} \tag{A7}$$

$$\lambda'_6\,\dot{\phi} = -\lambda_3 - 2r\dot{\phi}\lambda_4 + \frac{2\dot{r}}{r}\lambda_6 \tag{A8}$$

Putting (A5) in (A7), and collecting terms, yields

$$\lambda'_4 \dot{\phi} \dot{r} - \lambda_4 \ddot{r} - \lambda_3 \dot{\phi} + C - \lambda_6 \left(\frac{2\dot{r}\dot{\phi}}{r} + \ddot{\phi} \right) = 0 \tag{A9}$$

The solution to (A1) and (A2) is given by

$$r^2 \dot{\phi} = h = \sqrt{\mu \, p_c}$$

$$r = \frac{p_c}{1 + e_c \cos(\phi - \omega_c)}$$

We find $\dot{r}$ by

$$\dot{r} = \frac{r^2}{p_c} \dot{\phi} \, e_c \sin(\phi - \omega_c) = \frac{h \, e_c}{p_c} \sin(\phi - \omega_c)$$

From (A2), $\frac{2\dot{r}\dot{\phi}}{r} + \ddot{\phi} = 0$; thus (A9) becomes

$$\lambda'_4 - \frac{\lambda_4 \ddot{r}}{\dot{\phi}\dot{r}} = \frac{\lambda_3}{\dot{r}} - \frac{C}{\dot{\phi}\dot{r}} \tag{A10}$$

Defining true anomaly as $\theta \equiv \phi - \omega_c$, and using θ as the independent variable, we get upon substituting the equations of motion solution:

$$\frac{d\lambda_4}{d\theta} - \lambda_4 \cot\theta = \frac{\lambda_3 \, p}{h \, e \sin\theta} - \frac{C \, p^3}{h^2 \, e \sin\theta \, [1 + e\cos\theta]^2} \tag{A11}^*$$

where the subscript c is now omitted.

Substituting the orbit solution in equation (A8) we get

$$\frac{d\lambda_6}{d\theta} - \frac{\lambda_6 \, 2 \, e \sin\theta}{1 + e\cos\theta} = \frac{-\lambda_3 \, p^2}{h[1 + e\cos\theta]^2} - \frac{2 \, p \, \lambda_4(\theta)}{1 + e\cos\theta} \tag{A12}$$

---

*We note the singularity in this equation at $\theta = 0, \pi$, and that the limit approaches $\pm\infty$ on opposite sides of the singularity; the handling of this is discussed below.

We obtain the solution to (A11) first. The homogeneous equation is

$$\int \frac{d\lambda_4}{\lambda_4} = \int \cot\theta \, d\theta$$

$$\lambda_4 = K_1 \sin\theta \tag{A13}$$

Applying variation of constants, we insert (A13) into (A11), letting $K_1 = K_1(\theta)$.

$$K_1'(\theta) = \frac{C_1}{\sin^2\theta} - \frac{C_2}{\sin^2\theta\,[1 + e\cos\theta]^2}$$

where $C_1 \equiv \frac{\lambda_3 p}{he}$, $C_2 \equiv \frac{C p^3}{h^2 e}$

$$K_1(\theta) = -C_1 \cot\theta + C_3 - C_2 \int \frac{d\theta}{\sin^2\theta\,[1 + e\cos\theta]^2} \tag{A14}$$

Letting $u = [1 + e\cos\theta]^{-2}$, $dv = \csc^2\theta \, d\theta$, we get

$$\int u\,dv = -\cot\theta\,[1 + e\cos\theta]^{-2} + 2e\int \frac{\cos\theta\,d\theta}{[1 + e\cos\theta]^3}$$

Using ref. (10), we find

$$2e\int \frac{\cos\theta\,d\theta}{[1 + e\cos\theta]^3} = \frac{e}{(1 - e^2)}\left[\frac{\sin\theta}{(1 + e\cos\theta)^2} + \int \frac{[-2e + \cos\theta]\,d\theta}{[1 + e\cos\theta]^2}\right]$$

Multiple use of #317 and #309 in ref. (10) yields

$$\int \frac{d\theta}{(1 + e\cos\theta)^2} = \frac{1}{(1 - e^2)}\left[\frac{-e\sin\theta}{1 + e\cos\theta} + \frac{2}{\sqrt{1 - e^2}}\tan^{-1}\frac{\sqrt{1 - e^2}\,\tan\frac{1}{2}\theta}{1 + e}\right]$$

where $-\pi < \theta < \pi$ and $0 \leq e < 1$ - elliptical transfer orbits only.

Again, using #315 and #309 we obtain

$$\int \frac{\cos\theta\, d\theta}{(1 + e\cos\theta)^2}$$

Collecting terms we get:

$$\int \frac{d\theta}{\sin^2\theta\,[1 + e\cos\theta]^2} = -\cot\theta\,[1 + e\cos\theta]^{-2}$$

$$+ \frac{e}{(1 - e^2)}\left[\frac{\sin\theta}{(1 + e\cos\theta)^2} - \frac{2e}{(1 - e^2)}\left[\frac{-e\sin\theta}{1 + e\cos\theta}\right.\right.$$

$$\left. + \frac{2}{\sqrt{1 - e^2}}\tan^{-1}\frac{\sqrt{1 - e^2}\tan 1/2\,\theta}{1 + e}\right] + \frac{1}{1 - e^2}\left(\frac{\sin\theta}{1 + e\cos\theta}\right.$$

$$\left.\left. - \frac{2e}{\sqrt{1 - e^2}}\tan^{-1}\frac{\sqrt{1 - e^2}\tan 1/2\,\theta}{1 + e}\right)\right] + C_4 \equiv L + C_4$$

$$\therefore K_1(\theta) = -C_1\cot\theta + C_3 - C_2\left(-\cot\theta\,[1 + e\cos\theta]^{-2}\right.$$

$$+ \frac{e}{1 - e^2}\left[\frac{\sin\theta}{(1 + e\cos\theta)^2} + \frac{\sin\theta}{(1 - e^2)(1 + e\cos\theta)}[2e^2 + 1]\right.$$

$$\left.\left. - \frac{6e}{(1 - e^2)^{3/2}}\tan^{-1}(\text{ARG})\right] + C_4\right)$$

Defining the constant $C_3 - C_2 C_4 \equiv \overline{K}_1$, we have:

$$\lambda_4(\theta) = - C_1 \cos\theta + \bar{K}_1 \sin\theta - C_2 \sin\theta \left(- \cot\theta \, [1 + e\cos\theta]^{-2} \right.$$

$$+ \frac{e}{1 - e^2} \left[\frac{\sin\theta}{(1 + e\cos\theta)^2} + \frac{\sin\theta \, (2e^2 + 1)}{(1 - e^2)(1 + e\cos\theta)} \right.$$

$$\left.\left. - \frac{6e}{(1 - e^2)^{3/2}} \tan^{-1}(\mathrm{ARG}) \right] \right) \qquad \text{(A15)}$$

where

$$\mathrm{ARG} \equiv \frac{\sqrt{1 - e^2}\,\tan\theta/2}{1 + e},$$

and $\bar{K}_1$ is determined such that $\lambda_4(\phi_c - \omega_c) \equiv \lambda_4(\theta_c)$ is satisfied.

Turning now to equation (A12), we have for the homogeneous solution:

$$\lambda_6 = K_2 \, (1 + e\cos\theta)^{-2}$$

Using the form (A14) for $K_1(\theta)$ in the equation for $\lambda_4(\theta)$, substituting the homogeneous solution for $\lambda_6(\theta)$, above, into (A12) and considering that $K_2 = K_2(\theta)$, yields the differential equation for $K_2(\theta)$:

$$K_2'(\theta) \, [1 + e\cos\theta]^{-2} = - C_1' \, [1 + e\cos\theta]^{-2} - 2p \, [1 + e\cos\theta]^{-1}$$

$$\left[- C_1 \cos\theta + C_3 \sin\theta - C_2 \sin\theta \int \frac{d\theta}{\sin^2\theta \, [1 + e\cos\theta]^2} \right] ; \; C_1' \equiv \frac{\lambda_3 p^2}{h}$$

Thus,

$$K_2'(\theta) = - C_1' - 2p \, [1 + e\cos\theta] \left[- C_1 \cos\theta + C_3 \sin\theta \right.$$

$$\left. - C_2 \sin\theta \int \frac{d\theta}{\sin^2\theta \, [1 + e\cos\theta]^2} \right]$$

Now:

$$\int \left[- C_1' - 2 p [1 + e \cos \theta] \left(- C_1 \cos \theta + C_3 \sin \theta \right) \right] d \theta$$

$$= 2 p \left(C_1 \sin \theta + C_3 \cos \theta + \frac{e C_1}{2} \sin \theta \cos \theta - \frac{e C_3}{2} \sin^2 \theta\right) + C_5 \qquad \text{(A16)}$$

Finally, we need:

$$2 p C_2 \int \sin \theta [1 + e \cos \theta] \left(\int \frac{d \theta}{\sin^2 \theta [1 + e \cos \theta]^2} \right) d \theta \qquad \text{(A17)}$$

Let:

$$u = \int \frac{d \theta}{\sin^2 \theta [1 + e \cos \theta]^2} \; ; \qquad \begin{aligned} dv &= \sin \theta [1 + e \cos \theta] \, d \theta \\ v &= - \cos \theta + \frac{e}{2} \sin^2 \theta \end{aligned}$$

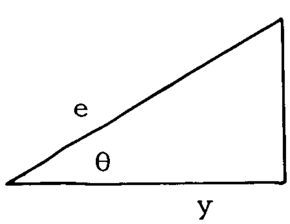
$$\int udv = uv - \int \frac{[- \cos \theta + e/2 \sin^2 \theta]}{\sin^2 \theta [1 + e \cos \theta]^2} d\theta$$

$$\int udv = uv + \int \frac{\cos \theta \, d \theta}{\sin^2 \theta [1 + e \cos \theta]^2} - \frac{e}{2} \int \frac{d \theta}{\left[1 + e \cos \theta\right]^2}$$

For the first integral let

$$y = e \cos \theta$$

$$dy = - e \sin \theta \, d \theta$$



$\sqrt{e^2 - y^2}$ $\quad \sin \theta = \dfrac{\sqrt{e^2 - y^2}}{e}$

$$\int \frac{\cos\theta\, d\theta}{\sin^2\theta\,[1 + e\cos\theta]^2} = -e \int \frac{y\, dy}{[1+y]^2\,(e^2 - y^2)^{3/2}}$$

Now, let

$$z = 1 + y;$$

$$y^2 = z^2 - 2z + 1$$

Then:

$$\int \frac{\cos\theta\, d\theta}{\sin^2\theta\,[1 + e\cos\theta]^2} = -e \int \frac{(z-1)\,dz}{z^2\, Z^{3/2}}$$

where

$$Z = -z^2 + 2z + e^2 - 1,$$

$$z = 1 + e\cos\theta$$

Using reference (10), #190 and #197:

$$\int \frac{\cos\theta\, d\theta}{\sin^2\theta\,[1 + e\cos\theta]^2} = -e \int \frac{dz}{z\, Z^{3/2}} + e \int \frac{dz}{z^2\, Z^{3/2}}$$

$$= \frac{e}{(e^2 - 1)}\left[\frac{-1}{Z^{1/2}}\left(1 + \frac{1}{z} + \frac{3}{e^2 - 1}\right) - \int \frac{dz}{z\, Z^{1/2}}\left(1 + \frac{3}{(e^2 - 1)}\right)\right.$$

$$\left. + 3\int \frac{dz}{Z^{3/2}}\left(1 + \frac{1}{e^2 - 1}\right)\right] + C_6 \qquad \text{(A18)}$$

where

$$\int \frac{dz}{z\, Z^{1/2}} = \frac{1}{\sqrt{1 - e^2}} \sin^{-1}\left(\frac{z + (e^2 - 1)}{z\, e}\right)$$

and

$$\int \frac{dz}{Z^{3/2}} = \frac{z - 1}{e^2\, Z^{1/2}}$$

Thus, from previous results in $\lambda_4(\theta)$ and collecting the above, we find:

$$K_2(\theta) = (A16) + 2\,p\,C_2\left(uv + (A18) - \frac{e}{2}\qquad \frac{1}{(1-e^2)}\left[\frac{-e\sin\theta}{1+e\cos\theta}\right.\right.$$

$$\left.\left. + \frac{2}{\sqrt{1-e^2}}\tan^{-1}\frac{\sqrt{1-e^2}\tan\theta/2}{1+e} + C_7\right]\right) \qquad \text{(A19)}$$

where u and v are defined below equation (A17).

Collecting terms, we get:

$$\lambda_6(\theta) = [1 + e\cos\theta]^{-2}\left[\overline{K}_2 + 2p(C_1\sin\theta + C_3\cos\theta + \frac{e\,C_1}{2}\sin\theta\cos\theta\right.$$

$$+ \frac{e\,C_3}{2}\sin^2\theta) + 2\,p\,C_2\left(\left[L + C_4\right]\left[-\cos\theta + \frac{e}{2}\sin^2\theta\right]\right.$$

$$+ \frac{e}{(e^2-1)}\left[-\frac{1}{Z^{1/2}}\left(1 + \frac{1}{z} + \frac{3}{e^2-1}\right) - \int\frac{dz}{z\,Z^{1/2}}\left(1 + \frac{3}{e^2-1}\right)\right.$$

$$\left. + 3\int\frac{dz}{Z^{3/2}}\left(1 + \frac{1}{e^2-1}\right)\right] + \frac{-e}{2(1-e^2)}\left(-\frac{e\sin\theta}{1+e\cos\theta}\right.$$

$$\left.\left.\left. + \frac{2}{\sqrt{1-e^2}}\tan^{-1}(\mathrm{ARG})\right)\right)\right] \qquad \text{(A20)}$$

We note that the constants C_3 and C_4 appear explicitly in (A20). To eliminate this, we consider all terms containing them, namely:

$$2\,p\left(C_3\cos\theta - \frac{e\,C_3}{2}\sin^2\theta\right) + 2\,p\,C_2\,C_4\left(-\cos\theta + \frac{e}{2}\sin^2\theta\right)$$

$$= C_3(2\,p\cos\theta - p\,e\sin^2\theta) - C_2\,C_4(2\,p\cos\theta - p\,e\sin^2\theta)$$

$$= \overline{K}_1\,p\,(2\cos\theta - e\sin^2\theta),$$

where $\overline{K}_1$ is the constant we determine from the initial conditions on $\lambda_4(\theta)$.

$$\therefore\ \lambda_6(\theta) = [1 + e\cos\theta]^{-2}\Bigg[\bar{K}_2 + 2pC_1 \sin\theta\left(1 + \frac{e\cos\theta}{2}\right)$$

$$+ \bar{K}_1\, p\,(2\cos\theta - e\sin^2\theta) + 2pC_2\Bigg(L\left(-\cos\theta + \frac{e}{2}\sin^2\theta\right)$$

$$+\frac{e}{e^2-1}\left[-\frac{1}{Z^{1/2}}\left(1 + \frac{1}{z} + \frac{3}{e^2-1}\right) - \int \frac{dz}{z\,Z^{1/2}}\left(1 + \frac{3}{e^2-1}\right)\right.$$

$$\left. + 3\int \frac{dz}{Z^{3/2}}\left(1 + \frac{1}{e^2-1}\right)\right] + \frac{-e}{2(1-e^2)}\left(\frac{-e\sin\theta}{1 + e\cos\theta}\right.$$

$$\left.\left.\left. + \frac{2}{\sqrt{1-e^2}}\tan^{-1}\left(\frac{\sqrt{1-e^2}\tan\theta/2}{1+e}\right)\right)\right)\right] \qquad \text{(A21)}$$

We note that equation (A11) has a singularity at $\theta = 0$ or π ($\phi = \omega_c$ or $\phi = \omega_c + \pi$). If it is necessary to evaluate λ_4 across either of these points, we have, from the first integral (A5), a solution.

$$\lim_{\substack{\phi \to \omega_c \\ \phi \to \omega_c + \pi}} \left[\begin{aligned} r &= \frac{p}{1 \pm e} \\ \dot{r} &= 0 \\ \dot{\phi} &= \frac{h}{p^2}(1 \pm e)^2 \\ \ddot{\phi} &= 0 \\ \ddot{r} &= \frac{\pm e\mu\,(1 \pm e)^2}{p^2} \end{aligned}\right.$$

where the upper sign is used for $\phi \to \omega_c$ ($\theta = 0$), and the lower for $\phi \to \omega_c + \pi$ ($\theta = \pi$).

We thus find, from (A5):

$$\lim_{\substack{\theta \to 0 \\ \theta \to \pi}} \quad \lambda_4 = \frac{C - \lambda_3 \phi}{\ddot{r}}$$

$$\lim_{\theta \to 0} \quad \lambda_4 = \frac{Cp^2}{e\,\mu(1+e)^2} - \frac{\lambda_3 h}{e\mu} \tag{A22}$$

We can derive (A22) in a different, and more fruitful manner. Rewrite equation (A11) as:

$$\frac{d\lambda_4}{d\theta} = \frac{1}{\sin\theta}\left[\lambda_4 \cos\theta + \frac{\lambda_3 p}{he} - \frac{Cp^3}{h^2 e\,[1 + e\cos\theta]^2}\right]$$

Since we require continuity of the multipliers, the bracketed quantity must approach zero just as $\sin\theta$ does as $\theta \to 0$. Solving, then, for λ_4 at $\theta = 0$, gives:

$$\lambda_4 = \frac{Cp^2}{e\mu\,(1+e)^2} - \frac{\lambda_3 h}{e\mu}$$

Thus, we know that

$$\lim_{\substack{\theta \to 0 \\ \theta \to \pi}} \quad \frac{d\lambda_4}{d\theta} \to \frac{0}{0}$$

We can thus use L'Hospital's Rule and derive two approximate differential equations for $\lambda_4(\theta)$. In the neighborhood of $\theta = 0$,

$$\frac{d\lambda_4}{d\theta} = -\lambda_4 \theta - \frac{2\,Cp^2\theta}{\mu\,(1+e)^3}$$

In the neighborhood of $\theta = \pi$,

$$\frac{d\lambda_4}{d\theta} = -\lambda_4\,(\theta - \pi) - \frac{2\,Cp^2\,(\theta - \pi)}{\mu\,(1-e)^3}$$

Solving these two equations, we obtain:

$$\underset{\theta \sim 0}{\lambda_4(\theta)} = -\frac{2\,Cp^2}{\mu\,(1+e)^3} + \bar{K}_3 \exp\left(-\frac{\theta^2}{2}\right)$$

$$\underset{\theta \sim \pi}{\lambda_4(\theta)} \doteq \frac{-2\,Cp^2\,(\theta - \pi)}{\mu(1-e)^3} + \bar{K}_4 \exp\left(-\theta\left[\frac{\theta}{2} - \pi\right]\right)$$

We can similarly approximate (A12), and obtain

$$\lambda_6(\theta)_{\theta \sim 0} = \exp\left(\frac{\theta^2 e}{1+e}\right)\left[\frac{\sqrt{\pi}}{2}\operatorname{erf}(\theta)\left(\sqrt{\frac{1+e}{e}}\left[-\frac{\lambda_3 p^2}{h(1+e)^2}+\frac{4Cp^3}{\mu(1+e)^4}\right]\right.\right.$$

$$\left.\left. - \frac{2p\,\bar{K}_3}{1+e}\sqrt{\frac{2(1+e)}{1+3e}}\right) + \bar{K}_5\right]$$

where erf (θ) is the error function, or probability integral:

$$\operatorname{erf}(\theta) = \frac{2}{\sqrt{\pi}}\int_0^{\theta} e^{-u^2}\,du$$

$$\lambda_6(\theta)_{\theta \sim \pi} = \exp\left(\frac{2e\,\theta}{1-e}\left[\pi - \frac{\theta}{2}\right]\right)\left[-\frac{\lambda_3 p^2 \theta}{h(1-e)^2}\exp\left(\frac{A_1\pi^2}{2}\right)\right.$$

$$\left[1+\frac{A_1\pi}{2}(\theta-\pi)\right] + \frac{Cp^3}{e\mu(1-e)^3}\exp\left(A_1\theta\left[\pi-\frac{\theta}{2}\right]\right) -$$

$$\left.\frac{2p\,\bar{K}_4\theta}{1-e}\exp\left(\frac{A_2\pi^2}{2}\right)\left[1+\frac{A_2\pi}{2}(\theta-\pi)\right] + \bar{K}_6\right]$$

where

$$A_1 = \frac{-2e}{1-e}$$

$$A_2 = \frac{1-3e}{1-e}$$

Since we do not have the switching function, k, as an explicit function of θ, some iterative method is needed to find the first θ at which k crosses from negative to positive values. Simply using two points and a slope to find a parabola for extrapolation works quite well. Writing k as:

$$k = \frac{c}{m}\left(\lambda_4 \cdot \frac{\lambda_4 r}{D_\nu} + \frac{\lambda_6}{r}\cdot\frac{\lambda_6}{D_\nu}\right) - \lambda_7$$

where

$$D_\nu = +\sqrt{\lambda_6^2 + \lambda_4^2 r^2}$$

and

$$\frac{d\,\lambda_7}{d\,\theta} = 0$$

we find

$$\frac{d\,k}{d\,\theta} = \frac{c}{m\,D_\nu}\left[\, r\,\lambda_4 \frac{d\,\lambda_4}{d\,\theta} + \frac{\lambda_6}{r}\;\frac{d\,\lambda_6}{d\,\theta} - \frac{\lambda_6^2\, e \sin\theta}{p}\right]$$

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OPTIMAL VARIABLE-THRUST RENDEZVOUS OF A POWER-LIMITED ROCKET BETWEEN NEIGHBORING LOW-ECCENTRICITY ORBITS

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Summary

A study has been made of minimum-fuel transfer and rendezvous between neighboring low-eccentricity orbits by power-limited rocket. This study includes and extends previous work wherein only the case of transfer between circular orbits was considered. As before, the analysis is based on the assumption that only small deviations from an initial orbit are allowed. Complete analytical solutions are obtained in three different sets of variables: (1) rotating rectangular coordinates, (2) rotating spherical coordinates, and (3) Lagrange's planetary variables. In addition to the determination of optimal transfer and rendezvous trajectories in three dimensions, synthesis of the optimal controls is also carried out in each case. The guidance coefficients resulting from the control synthesis are presented both in graphical form and in equation form suitable for use in guidance applications.

Introduction

It is characteristic of high-specific-impulse, low-thrust propulsion systems that the source of power is separate from the thrust device itself. Consequently, such propulsion systems are referred to as power-limited, since thrust is restricted in magnitude by the output of the power supply, which is in turn limited by the necessity of minimizing power supply weight.

The problem of transfer and rendezvous between neighboring orbits by a power-limited rocket is of interest for two basic reasons. First of all, the problem can be solved analytically, as was demonstrated in Refs. 1 and 2, provided that the thrust acceleration is not constrained in magnitude and that the proper simplifying assumptions are made in the mathematical model of the system. The analytic expressions thus obtained for the controls and for the optimum trajectories then provide insight into more general problems where the simplifying restrictions are lifted. Secondly, the solution to this problem provides a lower bound to the performance requirements for low-thrust orbital transfer and rendezvous.

It is interesting to note that if, for the same system model as has been used herein, the thrust acceleration is assumed constant, analytic integration of the equations of motion requires the evaluation of incomplete elliptic integrals of the third kind (Ref. 3). Therefore allowance for variable-thrust acceleration is essential if simple analytic solutions are to be obtained.

Analytical Method

Description of the Mathematical Model

The phrase "neighboring orbits," as defined here, requires that the inclination between orbit planes be small and that the radial separation between orbits be small relative to the semi-major axis of either orbit. If it is further assumed that motion in the transfer orbit does not deviate significantly from these neighboring orbits, linearization of the equations of motion is permissible.

The analysis has been carried out in three sets of variables: (1) rotating rectangular coordinates, (2) rotating spherical coordinates and, (3) Lagrange's planetary variables. The rotating coordinates have been utilized previously in Refs. 4, 5, 6 , while the planetary variables were applied to an orbit transfer problem in Ref. 3.

The rotating coordinate systems are depicted in Figs. 1 and 2. Each consists of an origin which revolves at satellite velocity in the initial (interior) circular orbit and orthogonal coordinates measured from this revolving origin. In the rectangular system of Fig. 1, y' is a radial dimension, x' is measured tangent to the initial orbit at the origin, and z' is a coordinate which is out of the plane of the initial orbit and is normal to both x' and y'.

In Fig. 2, the spherical system is composed of a radial coordinate y, an arc x, measured circumferentially from the origin, and another arc z, which is orthogonal to the x-y plane.

The Lagrange planetary variables, which are derived from the elements of an elliptic orbit and are used in the standard variation of parameters equations of celestial mechanics (Ref. 7), are convenient because they eliminate the necessity of treating singularities for zero eccentricity and zero inclination in these equations. As they are used in this study, the planetary variables consist of the non-dimensionalized semi-major axis $x_1 = a/a_0$, a circumferential distance component, x_4, and the following combinations of the remaining orbital elements:

$$\begin{aligned} x_2 &= e \sin \omega \\ x_3 &= e \cos \omega \\ x_5 &= \sin i \sin \Omega \\ x_6 &= \sin i \cos \Omega \end{aligned} \tag{1}$$

where e is eccentricity, ω is the longitude of peri-apsis, i is orbital inclination, and Ω is the longitude of the ascending node. The planetary variables provide a simple means of introducing eccentricity into the terminal orbits, and the form of the state equations using these variables is particularly simple in the present problem. However, in a practical application, they might be less desirable than the rotating coordinates because the orbital elements cannot be directly measured.

In view of the foregoing considerations, eccentric terminal orbits have been allowed only in the planetary variables in this study, while the analysis in rotating reference frames is confined to circular terminal orbits.

It should be noted here that the three sets of variables are entirely equivalent in that the equations of motion may be transformed directly from one set to another by substitution. There are some differences in the required linearizing assumptions which should be mentioned, however.

Consider the coordinate system depicted in Fig. 1, a rectangular system with its origin fixed on the interior orbit (assumed to be the reference orbit) in the x', y' plane. The mutually orthogonal coordinates x', y', z' form a triad that revolves with angular speed n_0 characteristic of the reference orbit, so that motion in this frame of reference is relative to a point on the reference orbit. The spherical coordinate system in Fig. 2 is described by the arc x in the plane of the reference orbit, the arc z measured normal to this plane, and a radial dimension y.

In order to linearize the equations of motion in the first system, it is necessary to assume that excursions x', y', z' from the origin be small in comparison with the radius, r_0, of the reference orbit. Motion is therefore constrained to a small sphere about the origin. No restrictions are placed on the component velocities. In the rotating spherical system, only the assumption of small component velocities will linearize the equations, whereas the arc x is not limited. The resultant motion is constrained to a torus about the reference orbit.

Since the linearized equations of motion are identical except for differences in notation (Ref. 4), one can draw the conclusion that, if in the spherical system the resultant motion does not involve large variations in x, the velocity components may be large. In the present study, use of the spherical system has been assumed throughout, and the results may be extended according to the foregoing discussion.

In the case of the planetary variables, the linearizing assumptions require that the difference in the semi-major axes of the terminal orbits be small and that the eccentricity of the terminal orbits as well the eccentricity of the instantaneous transfer orbit be small. The implications of these assumptions are similar to those for the rotating spherical system, in that "fast" trajectories are allowed only when the linearizing assumptions may be relaxed. On the other hand, fast trajectories are allowed in the rectangular system because no limits are placed on the component velocities in the linearizing process.

Analysis

The optimization problem is to derive the optimal control equation for the minimum-fuel transfer or rendezvous of a power-limited rocket between neighboring orbits in a given time. Mathematically, this requires minimization of the integral

$$J = \int_0^{t_f} (T/m)^2 dt = \int_0^{\tau_f} (n_o/2) A^2 d\tau = \int_0^{\tau_f} f_o(A) d\tau \qquad (2)$$

subject to constraints imposed by the equations of state which may be expressed in the form

$$\dot{x}_i = f_i \ (x, A) \qquad i = 1,\ldots,n \qquad (3)$$

The control is the thrust acceleration vector, A, in the present case.

The problem is treated as a problem of Lagrange in the calculus of variations. In particular, Breakwell's formulation (Ref. 8) of this problem is used because the linearized equations in the present case are particularly well suited to this formulation.

If a fundamental function F is defined as

$$F = -f_o + \sum_{i=1}^{n} \lambda_i f_i \qquad (4)$$

the variational treatment requires satisfaction of Euler-Lagrange equations in the following form as necessary conditions for the existence of an extremal arc:

$$\frac{d\lambda_i}{d\tau} = -\frac{\partial F}{\partial x_i} \qquad (5)$$

$$\frac{\partial F}{\partial A_j} = 0 \qquad (6)$$

An additional necessary condition provided by the Pontryagin Maximum Principle must also be satisfied to insure that the stationary solution predicted by the Euler equations is actually an extremum. The maximum principle, which may be expressed as

$$F\ (x_i, \lambda_i, A_j^*) \geq F\ (x_i, \lambda_i, A_j) \qquad (7)$$

ensures that the stationary solution is an absolute maximum. Furthermore, it has been shown (Ref. 9) that for a system where both the state variables and the controls appear linearly in the state equations, the maximum principle is also sufficient to ensure a minimum of the payoff, J. Since all cases in the present analyses are linear in the controls and satisfy the maximum principle, the optimum trajectories described herein are absolute extrema.

Due to the great number of equations involved, the variational analysis is not described in each case. Only the most important equations are included, and these are grouped in an orderly fashion in the appendices. The rotating coordinate systems are considered in Appendix I, and the planetary variables are considered in Appendix II. For a more detailed account of the application of the aforementioned equations the reader is referred to Ref. 1 wherein a specific case is treated in detail.

Synthesis of the Optimum Controls

In order to put the equations for the optimized controls into a form compatible with guidance requirements, several changes are made. First, τ in the control equations is replaced by $-\tau$. That is, the equations are rewritten with "time-to-go" as the independent variable. Secondly, while in the ordinary transfer and rendezvous analyses in rotating coordinates it was generally convenient to assume zero initial conditions, the terminals are reversed in the control synthesis. That is, the target orbit is assumed to be defined by zero values in most of the state variables. The results of the control synthesis are expressed in terms of the guidance coefficients, $\partial A_j/\partial x_i$, of each component of the control vector, A.

The equations for the control synthesis are summarized in Appendix III, for transfer and rendezvous in each of the coordinate systems. Those equations which deal specifically with transfer between circular orbits have been presented previously in Ref. 2.

Results

Orbit Transfer and Rendezvous

The multiplicity of solutions generated in this study (particularly for rendezvous) precludes a graphical presentation of all the resulting trajectories. An attempt is made to summarize the results in a reasonably concise form with orbit transfer solutions represented as special cases of rendezvous wherever feasible.

To simplify the presentation of the results, only circle-to-circle transfer and rendezvous cases are examined in the summary curves of Figs. 3 through 12. The first set of plots, Figs. 3 through 5, shows the variation of the components of the optimal thrust acceleration with time for circle-to-circle transfer only.

The in-plane components A_x/y_f and A_y/y_f are seen to display symmetry about the midpoint in time for all trip times, as does the out-of-plane component $A_z/r_0 i$. In particular, when $\tau_f = 2n\pi$, the components A_x/y_f and A_y/y_f are constant with time, and the latter is zero. For the coplanar problem, constant circumferential thrust acceleration is thereby specified as the optimum mode for integral multiples of the period of the reference orbit, a result that is in agreement with Ref. 6.

Figures 6 through 8 show the thrust acceleration components for circle-to-circle rendezvous at a particular trip time equal to one sixth of an orbital period of the reference orbit. The parameter in Figs. 6 and 7 is $x_f/y_f\tau_f$ which takes on the value 3/4 for the special case of optimum transfer. Similarly the out-of-plane component is plotted with Ω_f as a parameter. As indicated, the longitude of the node can have either of two values, 150 or 330 deg, for optimum transfer.

The payoff, J, can be best represented as the sum of three components, J_1, J_2, and J_3, which are defined by Eqs. A-44 and A-45 and are plotted in Figs. 9 through 11. The components J_1 and J_2 define propellant requirements for coplanar rendezvous, while the addition of J_3 introduces the out-of-plane requirement. In particular J is equal to J_1 for coplanar transfer since the term $x_f/y_f\tau_f - 3/4$ in J_2 is zero for optimum transfer.

All three components, as well as their sum, are seen to be monotonically decreasing functions of τ_f. In the limit, as $\tau_f \to \infty$, A and $J \to 0$. This is a consequence of the fact that no limit has been placed on exhaust velocity. Similarly all three components tend to infinity as τ_f approaches zero because zero trip time requires infinite thrust acceleration.

An interesting feature of J_3 is evident from Fig. 11. For $\tau_f = k\pi$ where $k = 0, 1, 2, \ldots$, J_3 is the same for all nodal longitudes, Ω_f. For all other times the envelope of the family of curves is given by the equations

$$J_{3max} = \frac{1}{\tau_f - |\sin \tau_f|} \tag{8}$$

$$J_{3min} = \frac{1}{\tau_f + |\sin \tau_f|} \tag{9}$$

where the lower envelope is given by Eq. 9 and represents J_3 for optimum transfer.

Application to Planetary Orbits

Strictly speaking, none of the planetary orbits are "neighboring orbits" in the sense in which this term has been defined. Earth's closest neighbor, Venus, has a semi-major axis, a = 0.7233 AU, compared with a = 1.0 AU for earth, leaving a separation distance of 0.2767 AU which is not << 1.0 AU. However, it is possible to apply the linearized analysis to earth-Venus trajectories with remarkably good accuracy. In Fig. 12 a comparison has been made with the exact solutions of Ref. 10 , for earth-Venus transfers. The circled points were calculated from Eq. A-43 of Appendix I. These results for the special case of uninclined circular terminal orbits show only a slight discrepancy in J for transfer times up to one earth year.

To obtain the circled points in Fig. 12 a reference orbit mid-way between the two terminal orbits was selected, i.e., a = 0.8617 AU. This improves the accuracy of the results over what could be obtained by referencing the coordinates to the major axis of either terminal orbit.

These results are encouraging and tend to support the view that an extension of the linearized analysis may be adequate for transfer and rendezvous between the orbits of earth and the nearby planets. Such an extension need not even be an exact second-order solution but might include only the dominant second-order terms in the equations of motion. This possibility is currently being explored by inclusion of the second-order terms in the radial motion.

Control Synthesis

In this study it has been possible to express each of the components of the optimal control vector, A, as a linear function of the n state variables.

$$A_j = \sum_{i=1}^{n} \frac{\partial A_j}{\partial x_i} x_i \qquad (10)$$

Therefore the presentation of the results can be confined to curves of the guidance coefficients, $\partial A_j/\partial x_i$ plotted against time to go, τ' . Using the equations for the guidance coefficients which comprise Appendix III, the summary curves of Figs. 13 through 24 were generated.

The synthesized controls for the case of transfer between an arbitrary state and a nearby circular orbit appear in Figs. 13 through 15 in terms of the rotating coordinate system variables. The extension to include eccentricity of the final orbit is provided by use of the Lagrange planetary variables in Figs. 16 through 18.

For rendezvous the same procedure is followed in the presentation of the synthesized controls, with the addition of curves to account for the dependence of in-plane thrust acceleration components on the circumferential distance. In rotating coordinates, Figs. 19 through 21 summarize the results for rendezvous between any initial state and a point on a nearby circular orbit.

As in the transfer case, the planetary variables facilitate the extension to rendezvous between an initial state and a point on a nearby orbit of low eccentricity. The results for the planetary variables appear in Figs. 22 through 24.

All the curves for the guidance coefficients display similar behavior. When time-to-go is short, the curves diverge to infinity, (either positive or negative), but a damped oscillation is evident, causing the coefficients to approach zero for very long times.

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<u>Nomenclature</u>

$\frac{T}{m}$	Thrust-to-mass ratio
A	$\frac{1}{n_0} \frac{T}{m}$
C	Integration constant
f	Rate of change of a state variable
F	Fundamental function
J	Defined by Eq. 2
D	Defined by Eq. A-146
B	Defined by Eq. A-174
Q	Defined by Eq. A-173
Φ	Defined by Eq A-138
λ	Lagrange multiplier
r	Radius
R	Radial force
W	Normal force
S	Circumferential force
n	Mean angular motion
x,y,z	Position components in spherical system
x',y',z'	Position components in rectangular system
u,v,w	Velocity components in x, y, z, directions
t	Time

Nomenclature (Contd.

τ	$n_o t$
τ'	Time to go
η	True anomaly
ω	Longitude of peri-apsis
e	Eccentricity
N	Unit vector normal to instantaneous transfer orbit
a	Semi-major axis
Ω	Longitude of the node
i	Inclination
x_1	a/a_o
x_2	$e \sin \omega$
x_3	$e \cos \omega$
x_5	$\sin i \sin \Omega$
x_6	$\sin i \cos \Omega$
$\vec{c}$	Angular momentum vector

Subscripts

i	Index denoting x,y,z,u,v,w
j	Index denoting x,y,z
o	Initial condition
f	Final condition
x,y,z,u,v,w	Denoting state variable

Nomenclature (Contd.

R	Radial
S	Circumferential
W	Normal

Superscripts

*	Optimum condition
$\rightarrow$	Denotes a vector

Appendix I

Rotating Rectangular and Spherical Coordinate Systems

1. Equations of State

$$\frac{dx}{d\tau} = u \qquad (A-1)$$

$$\frac{dy}{d\tau} = v \qquad (A-2)$$

$$\frac{dz}{d\tau} = w \qquad (A-3)$$

$$\frac{du}{d\tau} = A_x + 2y \qquad (A-4)$$

$$\frac{dv}{d\tau} = A_y + 3y - 2u \qquad (A-5)$$

$$\frac{dw}{d\tau} = A_z - z \qquad (A-6)$$

2. Euler-Lagrange Equations

$$\dot{\lambda}_x = 0 \qquad (A-7)$$

$$\dot{\lambda}_y = -3\lambda_v \qquad (A-8)$$

$$\dot{\lambda}_z = \lambda_w \qquad (A-9)$$

$$\dot{\lambda}_u = -\lambda_x + 2\lambda_v \qquad (A-10)$$

$$\dot{\lambda}_v = -\lambda_y - 2\lambda_u \qquad (A-11)$$

$$\dot{\lambda}_w = -\lambda_z \qquad (A-12)$$

$$\lambda_u = n_o A_x \qquad (A-13)$$

$$\lambda_v = n_o A_y \qquad (A-14)$$

$$\lambda_w = n_o A_z \qquad (A-15)$$

3. Integrated Euler-Lagrange Equations

$$\lambda_x = n_o C_o \tag{A-16}$$

$$\lambda_y = -6n_o(C_4 + C_o\tau - C_1\cos\tau + C_2\sin\tau) \tag{A-17}$$

$$\lambda_z = 2n_o(C_5\sin\tau + C_3\cos\tau) \tag{A-18}$$

$$\lambda_u = n_o(3C_4 + 3C_o\tau - 4C_1\cos\tau + 4C_2\sin\tau) \tag{A-19}$$

$$\lambda_v = 2n_o(C_o + C_1\sin\tau + C_2\cos\tau) \tag{A-20}$$

$$\lambda_w = 2n_o(C_5\cos\tau - C_3\sin\tau) \tag{A-21}$$

4. Boundary Conditions

State Variable	Transfer $\tau = 0$	Transfer $\tau = \tau_f$	Rendezvous $\tau = 0$	Rendezvous $\tau = \tau_f$
x	0	FREE	0	x_f
y	0	y_f	0	y_f
z	0	z_f	0	z_f
u	0	$\frac{3}{2}y_f$ (1)	0	$\frac{3}{2}y_f$ (1)
v	0	0	0	0
w	0	$\sqrt{r_o^2 i^2 - z_f^2}$ (2)	0	$\sqrt{r_o^2 i^2 - z_f^2}$ (2)

5. Integrated Equations of State - (with initial conditions)

$$x = \left[16(\tau - \sin\tau) - \frac{3}{2}\tau^3\right]C_o + \left[16(1-\cos\tau) - 10\tau\sin\tau\right]C_1 + \left[22\sin\tau - 10\tau\cos\tau - 12\tau\right]C_2 - \left[\frac{9}{2}\tau^2 - 12(1-\cos\tau)\right]C_4 \tag{A-22}$$

$$y = \left[8(1-\cos\tau) - 3\tau^2\right]C_o + 5\left[\sin\tau - \tau\cos\tau\right]C_1 + \left[5\tau\sin\tau - 8(1-\cos\tau)\right]C_2 + 6\left[\sin\tau - \tau\right]C_4 \tag{A-23}$$

$$z = \left[\tau\cos\tau - \sin\tau\right]C_3 + \left[\tau\sin\tau\right]C_5 \tag{A-24}$$

(1) REF 6
(2) REF 5

$$u = \left[16(1-\cos\tau) - \frac{9}{2}\tau^2\right]C_0 + \left[6\sin\tau - 10\tau\cos\tau\right] C_1 \qquad \text{(A-25)}$$

$$+ \left[10\tau\sin\tau - 12(1-\cos\tau)\right]C_2 + \left[12\sin\tau - 9\tau\right]C_4$$

$$v = \left[8\sin\tau - 6\tau\right]C_0 + \left[5\tau\sin\tau\right]C_1 + \left[5\tau\cos\tau - 3\sin\tau\right]C_2$$

$$+ 3\left[1-\cos\tau\right]C_4 \qquad \text{(A-26)}$$

$$w = \left[-\tau\sin\tau\right]C_3 + \left[\sin\tau + \tau\cos\tau\right]C_5 \qquad \text{(A-27)}$$

6. Transversality Conditions - Transfer

$$\lambda_x = C_0 = 0 \qquad \text{(A-28)}$$

$$\frac{C_5}{C_3} = \frac{\tan\tau_f + \frac{w_f}{z_f}}{1 - \frac{w_f}{z_f}\tan\tau_f} \qquad \text{(A-29)}$$

7. Constants of Integration
Transfer

$$C_1 = \frac{y_f\sin\tau_f}{16(1-\cos\tau_f) - \tau_f(5\tau_f + 3\sin\tau_f)} \qquad \text{(A-30)}$$

$$C_2 = \frac{-y_f(1-\cos\tau_f)}{16(1-\cos\tau_f) - \tau_f(5\tau_f + 3\sin\tau_f)} \qquad \text{(A-31)}$$

$$C_3 = \frac{(\sin\tau_f + \tau_f\cos\tau_f)z_f - (\tau_f\sin\tau_f)\sqrt{r_o^2 i^2 - z_f^2}}{\tau_f^2 - \sin^2\tau_f} \qquad \text{(A-32)}$$

$$C_4 = \frac{\frac{y_f}{6}(5\tau_f + 3\sin\tau_f)}{16(1-\cos\tau_f) - \tau_f(5\tau_f + 3\sin\tau_f)} \qquad \text{(A-33)}$$

Rendezvous

$$C_0 = \frac{\tau_f\, y_f \left(\frac{x_f}{y_f \tau_f} - \frac{3}{4} \right) (5\tau_f - 3\sin\tau_f)}{\frac{3}{4}\tau_f (5\tau_f - 3\sin\tau_f)(\tau_f^2 - 80) + 4(1-\cos\tau_f)(71\tau_f^2 - 64) + 248\tau_f^{\,2}\cos\tau_f} \quad \text{(A-34)}$$

$$C_1 = \frac{y_f \sin\tau_f}{16(1-\cos\tau_f) - \tau_f(5\tau_f + 3\sin\tau_f)} + C_0 \left[\frac{3\sin\tau_f - 8(1-\cos\tau_f)}{5\tau_f - 3\sin\tau_f} \right] \quad \text{(A-35)}$$

$$C_2 = \frac{-y_f(1-\cos\tau_f)}{16(1-\cos\tau_f) - \tau_f(5\tau_f + 3\sin\tau_f)} + C_0 \left[\frac{3\tau_f(1+\cos\tau_f) - 8\sin\tau_f}{5\tau_f - 3\sin\tau_f} \right] \quad \text{(A-36)}$$

$$C_3 = \frac{(\sin\tau_f + \tau_f\cos\tau_f)\, z_f - (\tau_f \sin\tau_f)\sqrt{r_0^{\,2}\, i^2 - z_f^{\,2}}}{(\tau_f^{\,2} - \sin^2\tau_f)} \quad \text{(A-37)}$$

$$C_4 = \frac{\frac{y_f}{6}(5\tau_f + 3\sin\tau_f)}{16(1-\cos\tau_f) - \tau_f(5\tau_f + 3\sin\tau_f)} - C_0 \frac{\tau_f}{2} \quad \text{(A-38)}$$

$$C_5 = \frac{(\tau_f \sin\tau_f)\, z_f + (\tau_f\cos\tau_f - \sin\tau_f)\sqrt{r_0^{\,2}\, i^2 - z_f^{\,2}}}{(\tau_f^2 - \sin^2\tau_f)} \quad \text{(A-39)}$$

8. Controls

$$A_x = 3C_4 + 3C_0\tau - 4C_1\cos\tau + 4C_2\sin\tau \quad \text{(A-40)}$$

$$A_y = 2\left[C_0 + C_1 \sin\tau + C_2\cos\tau \right] \quad \text{(A-41)}$$

$$A_z = 2\left[C_5\cos\tau - C_3\sin\tau \right] \quad \text{(A-42)}$$

9. Payoff

Transfer

$$\frac{J}{n_0^{\,3}\, r_0^{\,2}} = \frac{\left(\frac{y_f}{r_0} \right)^2 (5\tau_f + 3\sin\tau_f)}{8\left[\tau_f(5\tau_f + 3\sin\tau_f) - 16(1-\cos\tau_f) \right]} + \frac{i^2}{\tau_f + |\sin\tau_f|} \quad \text{(A-43)}$$

Rendezvous

$$\frac{J}{n_o^3 r_o^2} = J_1\left(\frac{y_f}{r_o}\right)^2 + J_2\left(\frac{y_f}{r_o}\right)^2\left(\frac{x_f}{y_f\tau_f} - \frac{3}{4}\right)^2 + J_3 i^2 \qquad \text{(A-44)}$$

$$\frac{J}{n_o^3 r_o^2} = \frac{\left(\frac{y_f}{r_o}\right)^2 (5\tau_f + 3\sin\tau_f)}{8\left[\tau_f(5\tau_f + 3\sin\tau_f) - 16(1-\cos\tau_f)\right]}$$

$$+ \frac{\frac{\tau_f^2}{2}\left(\frac{y_f}{r_o}\right)^2\left(\frac{x_f}{y_f\tau_f} - \frac{3}{4}\right)^2 (5\tau_f - 3\sin\tau_f)}{\frac{3}{4}\tau_f(5\tau_f - 3\sin\tau_f)(\tau_f^2 - 80) + 4(1-\cos\tau_f)(71\tau_f^2 - 64) + 248\tau_f^2\cos\tau_f}$$

$$+ i^2\left[\frac{\tau_f - \sin\tau_f\cos(2\Omega_f + \tau_f)}{(\tau_f^2 - \sin^2\tau_f)}\right] \qquad \text{(A-45)}$$

10. It should be pointed out that for each free end condition in the case of orbit transfer, the variational analysis predicts an optimum value for that particular state variable at the end point. In the rotating coordinate systems the x and z coordinates are left open at final time, τ_f . The end point for the optimal transfer is then determined in the analysis and is defined by the equations.

$$\left(\frac{z_f}{r_o}\right)^* = i\sqrt{\frac{1 \mp \cos\tau_f}{2}} \qquad \text{(A-46)}$$

$$\left(\frac{x_f}{y_f}\right)^* = \frac{3}{4}\tau_f \qquad \text{(A-47)}$$

Appendix II

Lagrange's Variables

In the theory of special perturbations, as derived in Ref. 7 for example, the equations for rates of change of the elements of an elliptic orbit are written in terms of the elements and acceleration components S, R, and W, which are perpendicular to the radius vector, radial and normal to the orbital plane, respectively.

Consider the five elements a, e, i, ω, Ω. The equations for small rates of change of these variables are

$$\frac{da}{dt} = \frac{2}{n\sqrt{1-e^2}}\left[eR\sin\eta + S(1 + e\cos\eta)\right] \tag{A-48}$$

$$\frac{de}{dt} = \frac{\sqrt{1-e^2}}{na}\left[R\sin\eta + \frac{2\cos\eta + e + e\cos^2\eta}{1+e\cos\eta} S \right] \tag{A-49}$$

$$\frac{di}{dt} = \frac{\sqrt{1-e^2}}{na} W\cos(\omega+\eta) \tag{A-50}$$

$$\frac{d\omega}{dt} = \frac{\sqrt{1-e^2}}{nae}\left[-R\cos\eta + \frac{2+e\cos\eta}{1+e\cos\eta} S\sin\eta - \frac{e\tan\frac{i}{2}\sin(\omega+\eta)}{1+e\cos\eta} W \right] \tag{A-51}$$

$$\frac{d\Omega}{dt} = \frac{\sqrt{1-e^2}}{na}\frac{W}{\sin i}\sin(\omega+\eta) \tag{A-52}$$

In order to avoid singularities for zero eccentricity and inclination in Eqs. A-51 and A-52 these equations may be transformed according to the following definitions:

$$x_2 = e\sin\omega \tag{A-53}$$

$$x_3 = e\cos\omega \tag{A-54}$$

$$x_5 = \sin i\sin\Omega \tag{A-55}$$

$$x_6 = \sin i\cos\Omega \tag{A-56}$$

Under the assumptions

$$
\begin{aligned}
e &<< 1 \\
a &\approx a_0 \\
n &\approx n_0 \\
\tau = n_0 t &= \omega + \eta \\
i &<< 1
\end{aligned}
\tag{A-57}
$$

$$A_R = \frac{R}{a_0 n_0^2} \; , \quad A_S = \frac{S}{a_0 n_0^2} \; , \quad A_W = \frac{W}{a_0 n_0^2} \tag{A-58}$$

and with the further definitions

$$x_1 = \frac{a}{a_0} \tag{A-59}$$

$$x_4 = x \tag{A-60}$$

the equations of state for the variational problem may be derived from Eqs. A-48 through A-56 .

There is a direct equivalence between these equations and the equations of state in the rotating coordinate system variables. That is, each of the Lagrange variables x_1, x_2, x_3, ... x_6, can be expressed in terms of the rotating coordinate variables, x, y, z, u, v, and w.

Referring to Fig. 25 , define a position vector $\vec{r}$ in nonrotating coordinates originating at the center of attraction F. Assume the motion out of the reference plane is uncoupled from the in-plane motion.

Relative to a rotating rectangular coordinate system originating at O and rotating with angular velocity $\vec{n}$ this vector is

$$\vec{r} = x\vec{i} + (r_0 + y)\vec{j} \tag{A-61}$$

where the unit vectors $\vec{i}$ and $\vec{j}$ are taken in the x and y directions, respectively. The vector velocity $\vec{V}$ is obtained by differentiating $\vec{r}$.

$$\vec{V} = \frac{d\vec{r}}{dt} = u\vec{i} + v\vec{j} + \vec{n} \times \vec{r} \tag{A-62}$$

Since $\vec{n} = n_o \vec{k}$, the expression for $\vec{V}$ is

$$\vec{V} = \left[u - n_o (r_o + y) \right] \vec{i} + (v + n_o x)\, \vec{j} \tag{A-63}$$

Using Eqs. A-61 and A-63, expressions can be written for the angular momentum $\vec{C}$, the path speed V and the radius r of the vehicle

$$\vec{C} = \vec{r} \times \vec{V} = \left[x(v + n_o x) - (r_o + y)(u - n_o(r_o + y)) \right] \vec{k} \tag{A-64}$$

$$V = \sqrt{\vec{V} \cdot \vec{V}} = \sqrt{\left[u - n_o (r_o + y) \right]^2 + \left[v + n_o x \right]^2} \tag{A-65}$$

$$r = \sqrt{\vec{r} \cdot \vec{r}} = \sqrt{x^2 + (r_o + y)^2} \tag{A-66}$$

The following equations can be written for the angular momentum, speed and radius of a body in an inverse square field.

$$|\vec{C}| = \sqrt{Ka(1 - e^2)} \tag{A-67}$$

$$V = \sqrt{K\left(\frac{2}{r} - \frac{1}{a}\right)} = \sqrt{(\dot{r})^2 + (r\dot{\eta})^2} \tag{A-68}$$

$$r = \frac{a(1 - e^2)}{1 + e \cos \eta} \tag{A-69}$$

Combining these equations with the absolute value of $\vec{C}$, and with V and r from Eqs. A-64, A-65 and A-66 the following scalar equations result.

$$\frac{a}{a_o} = \left(1 + \frac{y}{r_o}\right)(1 + e \cos \eta) \tag{A-70}$$

$$\frac{u}{n_o r_o} - \left(1 + \frac{y}{r_o}\right) = \frac{\sqrt{\frac{a}{a_o}}}{1 + \frac{y}{r_o}} \tag{A-71}$$

$$\frac{v}{n_o r_o} + \frac{x}{r_o} = \sqrt{\frac{e \cos \eta}{\frac{a}{a_o}}} \tag{A-72}$$

Finally, noting that

$$\frac{a}{a_0} = x_1 \ , \quad x_2 = e \sin\omega \ , \quad x_3 = e \cos\omega$$
$$e \cos\eta = e\cos(\tau - \omega) = x_2 \sin\tau + x_3 \cos\tau \tag{A-73}$$

the equations relating the coordinates are obtained.

$$\frac{y}{r_0} = (x_1 - 1) - x_2 \sin\tau - x_3 \cos\tau \tag{A-74}$$

$$\frac{v}{n_0 r_0} = x_3 \cos\tau - x_2 \sin\tau \tag{A-75}$$

$$\frac{u}{n_0 r_0} = \frac{3}{2}(x_1 - 1) - 2x_2 \sin\tau - 2x_3 \cos\tau \tag{A-76}$$

The components of the out-of-plane motion can be related in the following way. If $\vec{N}$ is a unit vector normal to the instantaneous transfer orbit and $\vec{s}$ is a unit vector in the direction of the line of nodes, then

$$\vec{s} = \vec{N} \times \vec{k} \tag{A-77}$$

and, since the angle between $\vec{s}$ and the vehicle is $\tau - \Omega$

$$\cos(\tau - \Omega) = \vec{s} \cdot \vec{i} \tag{A-78}$$

Also, the orbital inclination is

$$\cos i = \vec{N} \cdot \vec{k} \tag{A-79}$$

Using these parameters the equation for the elevation, z, of the probe is

$$\frac{z}{r_0} = \tan i \sin(\tau - \Omega) \simeq \sin i \sin(\tau - \Omega) \tag{A-80}$$

or

$$\frac{z}{r_0} = - x_5 \cos\tau + x_6 \sin\tau \tag{A-81}$$

The out-of-plane velocity, w, is

$$\frac{w}{n_o r_o} = x_5 \sin\tau + x_6 \cos\tau \qquad \text{(A-82)}$$

1. Equations of State

$$\frac{dx_1}{d\tau} = 2A_S \qquad \text{(A-83)}$$

$$\frac{dx_2}{d\tau} = 2A_S \sin\tau - A_R \cos\tau \qquad \text{(A-84)}$$

$$\frac{dx_3}{d\tau} = 2A_S \cos\tau + A_R \sin\tau \qquad \text{(A-85)}$$

$$\frac{dx_4}{d\tau} = \frac{3}{2}(x_1 - 1) - 2x_2 \sin\tau - 2x_3 \cos\tau \qquad \text{(A-86)}$$

$$\frac{dx_5}{d\tau} = -A_W \sin\tau \qquad \text{(A-87)}$$

$$\frac{dx_6}{d\tau} = A_W \cos\tau \qquad \text{(A-88)}$$

2. Euler-Lagrange Equations

$$\dot{\lambda}_1 = -\frac{3}{2}\lambda_4 \qquad \text{(A-89)}$$

$$\dot{\lambda}_2 = 2\lambda_4 \sin\tau \qquad \text{(A-90)}$$

$$\dot{\lambda}_3 = 2\lambda_4 \cos\tau \qquad \text{(A-91)}$$

$$\dot{\lambda}_4 = \dot{\lambda}_5 = \dot{\lambda}_6 = 0 \qquad \text{(A-92)}$$

$$n_o A_S = 2(\lambda_1 + \lambda_2 \sin\tau + \lambda_3 \cos\tau) \qquad \text{(A-93)}$$

$$n_o A_R = -\lambda_2 \cos\tau + \lambda_3 \sin\tau \qquad \text{(A-94)}$$

$$n_o A_W = -\lambda_5 \sin\tau + \lambda_6 \cos\tau \qquad \text{(A-95)}$$

3. Integrated Euler-Lagrange Equations

$$\lambda_1 = \lambda_{10} - \frac{3}{2}\lambda_4 \tau \qquad \text{(A-96)}$$

$$\lambda_2 = \lambda_{20} - 2\lambda_4 \cos\tau \qquad \text{(A-97)}$$

$$\lambda_3 = \lambda_{30} + 2\lambda_4 \sin\tau \qquad \text{(A-98)}$$

$$\lambda_4 = \text{CONSTANT} \qquad \text{(A-99)}$$

$$\lambda_5 = \text{"} \qquad \text{(A-100)}$$

$$\lambda_6 = \text{"} \qquad \text{(A-101)}$$

4. Boundary Conditions

A great simplification in the complexity of the equations can be achieved by taking advantage of the symmetry afforded by the Lagrange variables x_2 and x_3. Therefore, in performing the integrations it will be convenient to use limits $-\tau_f/2$ to $\tau_f/2$ for the "in-plane" state variables.

	Transfer		Rendezvous	
State Variable ("in-plane")	$\tau = -\frac{\tau_f}{2}$	$\tau = \frac{\tau_f}{2}$	$\tau = -\frac{\tau_f}{2}$	$\tau = \frac{\tau_f}{2}$
x_1	1	$\Delta x_{1_f} + 1$	1	$\Delta x_{1_f} + 1$
x_2	x_{20}	$x_{20} + \Delta x_{2_f}$	x_{20}	$x_{20} + \Delta x_{2_f}$
x_3	x_{30}	$x_{30} + \Delta x_{3_f}$	x_{30}	$x_{30} + \Delta x_{3_f}$
x_4	x_{40}	FREE	x_{40}	$x_{40} + \Delta x_{4_f}$
(out-of-plane)	$\tau = 0$	$\tau = \tau_f$	$\tau = 0$	$\tau = \tau_f$
x_5	0	x_{5_f}	0	x_{5_f}
x_6	0	x_{6_f}	0	x_{6_f}

5. Integrated Equations of State (with initial conditions)

$$\Delta x_1 = 4\lambda_{10}(\tau + \frac{\tau_f}{2}) - 4\lambda_{20}(\cos\tau - \cos\frac{\tau_f}{2}) + 4\lambda_{30}(\sin\tau + \sin\frac{\tau_f}{2}) - 3\lambda_4(\tau^2 - \frac{\tau_f^2}{4})$$

$$\Delta x_2 = -4\lambda_{10}(\cos\tau - \cos\frac{\tau_f}{2}) + \frac{\lambda_{20}}{2}\left[5(\tau + \frac{\tau_f}{2}) - 3(\sin\tau\cos\tau + \frac{\sin\tau_f}{2})\right] \quad \text{(A-102)}$$

$$+ \frac{3}{2}\lambda_{30}(\sin^2\tau - \sin^2\frac{\tau_f}{2}) - 2\lambda_4\left[4(\sin\tau + \sin\frac{\tau_f}{2}) - 3(\tau\cos\tau + \frac{\tau_f}{2}\cos\frac{\tau_f}{2})\right] \quad \text{(A-103)}$$

$$\Delta x_3 = 4\lambda_{10}(\sin\tau + \sin\frac{\tau_f}{2}) + \frac{3}{2}\lambda_{20}(\sin^2\tau - \sin^2\frac{\tau_f}{2})$$

$$+ \frac{\lambda_{30}}{2}\left[5(\tau + \frac{\tau_f}{2}) + 3(\sin\tau\cos\tau + \frac{\sin\tau_f}{2})\right]$$

$$-2\lambda_4\left[4(\cos\tau - \frac{\cos\tau_f}{2}) + 3(\tau\cos\tau + \frac{\tau_f}{2}\cos\frac{\tau_f}{2})\right] \quad \text{(A-104)}$$

$$\Delta x_4 = \lambda_{10}\left\{3(\tau + \frac{\tau_f}{2})^2 - 8\left[1 - \cos(\tau + \frac{\tau_f}{2})\right]\right\}$$

$$+\lambda_{20}\left\{(\tau + \frac{\tau_f}{2})\left[5\cos\tau + 6\cos\frac{\tau_f}{2}\right] - \frac{3}{2}\sin(\tau + \frac{\tau_f}{2}) - \frac{19}{2}\sin\tau - 8\sin\frac{\tau_f}{2}\right\}$$

$$+\lambda_{30}\left\{(\tau + \frac{\tau_f}{2})\left[6\sin\frac{\tau_f}{2} - 5\sin\tau\right] + \frac{3}{2}\cos(\tau + \frac{\tau_f}{2}) - \frac{19}{2}\cos\tau + 8\cos\frac{\tau_f}{2}\right\}$$

$$+\lambda_4\left\{16(\tau + \frac{\tau_f}{2}) - 6\tau_f\left[1 - \cos(\tau + \frac{\tau_f}{2})\right] + 3\left(\frac{\tau_f}{2}\right)^3 + \frac{9}{2}\tau\left(\frac{\tau_f}{2}\right)^2 - \frac{3}{2}\tau^3\right\} \quad \text{(A-105)}$$

$$+ 2x_{20}\left[\cos\tau - \cos\frac{\tau_f}{2}\right] - 2x_{30}\left[\sin\tau + \sin\frac{\tau_f}{2}\right]$$

$$x_5 = \frac{\lambda_5}{2}(\tau - \sin\tau\cos\tau) - \frac{\lambda_6}{2}\sin^2\tau \quad \text{(A-106)}$$

$$x_6 = -\frac{\lambda_5}{2}\sin^2\tau + \frac{\lambda_6}{2}(\tau + \sin\tau\cos\tau) \quad \text{(A-107)}$$

6. Transversality Conditions - Transfer

$$\lambda_4 = 0 \quad \text{(A-108)}$$

$$\frac{\lambda_5}{\lambda_6} = \tan\tau \quad \text{(A-109)}$$

7. Constants of Integration

Transfer

$$\lambda_{10} = \frac{\frac{\Delta x_{1f}}{4}(5\tau_f + 3\sin\tau_f) - 4\Delta x_{3f}\sin\frac{\tau_f}{2}}{\tau_f(5\tau_f + 3\sin\tau_f) - 16(1 - \cos\tau_f)} \quad \text{(A-110)}$$

$$\lambda_{20} = \frac{2\,\Delta x_{2f}}{5\tau_f - 3\sin\tau_f} \tag{A-111}$$

$$\lambda_{30} = \frac{2\left[\tau_f \Delta x_{3f} - 2\Delta x_{1f}\sin\frac{\tau_f}{2}\right]}{\tau_f(5\tau_f + 3\sin\tau_f) - 16(1-\cos\tau_f)} \tag{A-112}$$

$$\lambda_5 = \frac{x_{5f}(\tau_f + \sin\tau_f\cos\tau) + x_{6f}\sin^2\tau_f}{2(\tau_f^2 - \sin^2\tau_f)} \tag{A-113}$$

<u>Rendez vous</u>

$$\lambda_{10} = \frac{\frac{\Delta x_{1f}}{4}(5\tau_f + 3\sin\tau_f) - 4\Delta x_{3f}\sin\frac{\tau_f}{2}}{\tau_f(5\tau_f + 3\sin\tau_f) - 16(1-\cos\tau_f)} \tag{A-114}$$

$$\begin{aligned}\lambda_{20} = {} & \frac{1}{\tau_f(5\tau_f - 3\sin\tau_f)\left(\frac{3}{16}\tau_f^2 + 1\right) - 2\left(8\sin\frac{\tau_f}{2} - 3\tau_f\cos\frac{\tau_f}{2}\right)^2}\Bigg[\frac{3}{4}\tau_f\Delta x_{1f}\left(3\tau_f\cos\frac{\tau_f}{2} - 8\sin\frac{\tau_f}{2}\right) \\ & + \Delta x_{2f}\left[\frac{3}{8}\tau_f^3 + 8\tau_f - 3\tau_f(1-\cos\tau_f) - 8\sin\tau_f\right] \\ & - \left[3\tau_f\cos\frac{\tau_f}{2} - 8\sin\frac{\tau_f}{2}\right]\left[2\Delta x_{3f}\sin\frac{\tau_f}{2} + \Delta x_{4f} + 4x_{30}\sin\frac{\tau_f}{2}\right]\Bigg]\end{aligned} \tag{A-115}$$

$$\lambda_{30} = \frac{2\left[\tau_f\,\Delta x_{3f} - 2\Delta x_{1f}\sin\frac{\tau_f}{2}\right]}{\tau_f(5\tau_f + 3\sin\tau_f) - 16(1-\cos\tau_f)} \tag{A-116}$$

$$\begin{aligned}\lambda_4 = {} & \frac{1}{\tau_f(5\tau_f - 3\sin\tau_f)\left(\frac{3}{16}\tau_f^2 + 1\right) - 2\left(8\sin\frac{\tau_f}{2} - 3\tau_f\cos\frac{\tau_f}{2}\right)^2}\Bigg[-\frac{3}{16}\tau_f\Delta x_{1f}(5\tau_f - 3\sin\tau_f) \\ & - \frac{\Delta x_{2f}}{2}\left[11\tau_f\cos\frac{\tau_f}{2} + 3\sin\frac{\tau_f}{2}(1-\cos\tau_f) - 22\sin\frac{\tau_f}{2}\right] \\ & + (5\tau_f - 3\sin\tau_f)\left[\frac{\Delta x_{3f}}{2}\sin\frac{\tau_f}{2} + \frac{\Delta x_{4f}}{4} + x_{30}\sin\frac{\tau_f}{2}\right]\Bigg]\end{aligned} \tag{A-117}$$

$$\lambda_5 = \frac{2\left\{x_{5f}(\tau_f + \sin\tau_f\cos\tau_f) + x_{6f}\sin^2\tau_f\right\}}{\tau_f^2 - \sin^2\tau_f} = \frac{2i\left[\tau_f\sin\Omega_f + \sin\tau_f\sin(\Omega_f+\tau_f)\right]}{\tau_f^2 - \sin^2\tau_f} \tag{A-118}$$

$$\lambda_6^{\bullet} = \frac{2\left\{x_{5f}\sin^2\tau_f + x_{6f}(\tau_f - \sin\tau_f\cos\tau_f)\right\}}{\tau_f^2 - \sin^2\tau_f} = \frac{2i\left[\tau_f\cos\Omega_f - \sin\tau_f\cos(\Omega_f+\tau_f)\right]}{\tau_f^2 - \sin^2\tau_f} \tag{A-119}$$

8. Controls

$$n_o A_S = 2\lambda_{10} - 3\lambda_4\tau + 2\lambda_{20}\sin\tau + 2\lambda_{30}\cos\tau \tag{A-120}$$

$$n_o A_R = 2\lambda_4 - \lambda_{20}\cos\tau + \lambda_{30}\sin\tau \tag{A-121}$$

$$n_o A_W = -\lambda_5\sin\tau + \lambda_6\cos\tau \tag{A-122}$$

9. Payoff

Transfer

$$\frac{J}{n_o^3 r_o^2} = \frac{\frac{\Delta x_{1f}^2}{8}(5\tau_f + 3\sin\tau_f) - 4\Delta x_{1f}\Delta x_{3f}\sin\frac{\tau_f}{2} + \tau_f\Delta x_{3f}^2}{\tau_f(5\tau_f + 3\sin\tau_f) - 16(1-\cos\tau_f)} \tag{A-123}$$

$$+ \frac{2\Delta x_{2f}^2}{5\tau_f - 3\sin\tau_f} + \frac{i^2}{\tau_f + |\sin\tau_f|}$$

Rendezvous

$$\frac{J}{n_o^3 r_o^2} = \frac{\frac{\Delta x_{1f}^2}{8}(5\tau_f + 3\sin\tau_f) - 4\Delta x_{1f}\Delta x_{3f}\sin\frac{\tau_f}{2} + \tau_f \Delta x_{3f}^2}{\tau_f(5\tau_f + 3\sin\tau_f) - 16(1-\cos\tau_f)}$$

$$+ \frac{\frac{1}{8}(5\tau_f - 3\sin\tau_f)\left\{2\Delta x_{2f}\cos\frac{\tau_f}{2} - 2\Delta x_{3f}\sin\frac{\tau_f}{2} - \Delta x_{4f} + \frac{3}{4}\tau_f\Delta x_{1f} - 4x_{30}\sin\frac{\tau_f}{2}\right\}^2}{\tau_f(5\tau_f - 3\sin\tau_f)(\frac{3}{16}\tau_f^2 + 1) - 2(3\tau_f\cos\frac{\tau_f}{2} - 8\sin\frac{\tau_f}{2})^2}$$

$$+ \frac{\Delta x_{2f}(3\tau_f\cos\frac{\tau_f}{2} - 8\sin\frac{\tau_f}{2})\left\{2\Delta x_{2f}\cos\frac{\tau_f}{2} - 2\Delta x_{3f}\sin\frac{\tau_f}{2} - \Delta x_{4f} + \frac{3}{4}\tau_f\Delta x_{1f} - 4x_{30}\sin\frac{\tau_f}{2}\right\}}{\tau_f(5\tau_f - 3\sin\tau_f)(\frac{3}{16}\tau_f^2 + 1) - 2(3\tau_f\cos\frac{\tau_f}{2} - 8\sin\frac{\tau_f}{2})^2}$$

$$+ \frac{\tau_f\Delta x_{2f}^2(\frac{3}{16}\tau_f^2 + 1)}{\tau_f(5\tau_f - 3\sin\tau_f)(\frac{3}{16}\tau_f^2 + 1) - 2(3\tau_f\cos\frac{\tau_f}{2} - 8\sin\frac{\tau_f}{2})^2}$$

$$+ i^2\left[\frac{\tau_f - \sin\tau_f\cos(2\Omega_f + \tau_f)}{(\tau_f^2 - \sin^2\tau_f)}\right] \quad \text{(A-124)}$$

10. The optimal values for changes in the state variables x_4 and Ω are predicted by the variational analysis in the case of orbit transfer where the values x_4 and Ω are left open at the final time.

$$\Delta x_4^* = \frac{3}{4}\tau_f\Delta x_{1f} - 2\Delta x_{3f}\sin\frac{\tau_f}{2} - 4x_{30}\sin\frac{\tau_f}{2}$$

$$+ \frac{4\Delta x_{2f}}{5\tau_f - 3\sin\tau_f}\left\{\frac{1}{2}\cos\frac{\tau_f}{2}(5\tau_f - 3\sin\tau_f) + 3\tau_f\cos\frac{\tau_f}{2} - 8\sin\frac{\tau_f}{2}\right\} \quad \text{(A-125)}$$

$$\Omega_f^* = n\pi - \frac{\tau_f}{2} \quad \text{(A-126)}$$

Appendix III

Synthesis of the Optimal Controls

A. Rotating Coordinates

1. Control Equations

$$A_y = \frac{\partial A_y}{\partial y} y + \frac{\partial A_y}{\partial u} u + \frac{\partial A_y}{\partial v} v + \frac{\partial A_y}{\partial x} x \qquad \text{(A-127)}$$

$$A_x = \frac{\partial A_x}{\partial y} y + \frac{\partial A_x}{\partial u} u + \frac{\partial A_x}{\partial v} v + \frac{\partial A_x}{\partial x} x \qquad \text{(A-128)}$$

$$A_z = \frac{\partial A_z}{\partial z} z + \frac{\partial A_z}{\partial w} w \qquad \text{(A-129)}$$

2. Guidance Coefficients-Transfer

$$\frac{\partial A_y}{\partial y} = \frac{12\tau'}{\Phi}(1 - \cos\tau')(29 - 27\cos\tau') \qquad \text{(A-130)}$$

$$\frac{\partial A_y}{\partial u} = \frac{24}{\Phi}(1 - \cos\tau')(11\sin\tau' - 3\tau'\cos\tau' - 8\tau') \qquad \text{(A-131)}$$

$$\frac{\partial A_y}{\partial v} = \frac{12}{\Phi}(5\tau'^2 + 3\tau'\sin\tau'\cos\tau' - 8\sin^2\tau') \qquad \text{(A-132)}$$

$$\frac{\partial A_x}{\partial y} = \frac{12}{\Phi}\left[70\tau'\sin\tau' - 55\tau'^2 + 18\tau'\sin\tau'\cos\tau' + 3(1 - \cos\tau')(5 - 27\cos\tau')\right] \qquad \text{(A-133)}$$

$$\frac{\partial A_x}{\partial u} = \frac{6}{\Phi}\left[65\tau'^2 - 80\tau'\sin\tau' - 24\tau'\sin\tau'\cos\tau' - (1 - \cos\tau')(25 - 103\cos\tau')\right] \qquad \text{(A-134)}$$

$$\frac{\partial A_x}{\partial v} = -\frac{24}{\Phi}\left(8\tau' - 11\sin\tau' + 3\tau'\cos\tau'\right)\left(1-\cos\tau'\right)^{(1)} \tag{A-135}$$

$$\frac{\partial A_z}{\partial z} = \frac{-2\sin^2\tau'}{\tau'^2 - \sin^2\tau'} \tag{A-136}$$

$$\frac{\partial A_z}{\partial w} = \frac{-(2\tau' - \sin 2\tau')}{\tau'^2 - \sin^2\tau'} \tag{A-137}$$

where

$$\Phi = 480\tau' - 75\tau'^3 - 240\tau'\cos\tau'(1+\cos\tau') - 144\sin\tau'(1-\cos\tau') - 213\tau'\sin^2\tau' \tag{A-138}$$

3. Rendezvous

Due to the length and complexity of the synthesized, in-plane, control equations for rendezvous, the guidance coefficients are not written explicitly here. Instead the basic equations are tabulated, and the coefficients calculated from these equations are plotted in Fig. 19 through 21.

$$\frac{\partial A_x}{\partial x_i} = 3\frac{\partial C_4}{\partial x_i} - 3\frac{\partial C_0}{\partial x_i}\tau' - 4\frac{\partial C_1}{\partial x_i}\cos\tau' - 4\frac{\partial C_2}{\partial x_i}\sin\tau' \tag{A-139}$$

$$\frac{\partial A_y}{\partial x_i} = 2\left(\frac{\partial C_0}{\partial x_i} - \frac{\partial C_1}{\partial x_i}\sin\tau' + \frac{\partial C_2}{\partial x_i}\cos\tau'\right) \tag{A-140}$$

$$A_z = \frac{\partial A_z}{\partial z}z + \frac{\partial A_z}{\partial w}w \tag{A-141}$$

(1) NOTE : $\frac{\partial A_x}{\partial v} = \frac{\partial A_y}{\partial u}$

$$C_0 = \frac{\begin{vmatrix} x & \phi_{11} & \phi_{12} & \phi_{14} \\ y & \phi_{21} & \phi_{22} & \phi_{24} \\ u & \phi_{31} & \phi_{32} & \phi_{34} \\ v & \phi_{41} & \phi_{42} & \phi_{44} \end{vmatrix}}{D} \qquad \text{(A-142)}$$

$$C_1 = \frac{\begin{vmatrix} \phi_{10} & x & \phi_{12} & \phi_{14} \\ \phi_{20} & y & \phi_{22} & \phi_{24} \\ \phi_{30} & u & \phi_{32} & \phi_{34} \\ \phi_{40} & v & \phi_{42} & \phi_{44} \end{vmatrix}}{D} \qquad \text{(A-143)}$$

$$C_2 = \frac{\begin{vmatrix} \phi_{10} & \phi_{11} & x & \phi_{14} \\ \phi_{20} & \phi_{21} & y & \phi_{24} \\ \phi_{30} & \phi_{31} & u & \phi_{34} \\ \phi_{40} & \phi_{41} & v & \phi_{44} \end{vmatrix}}{D} \qquad \text{(A-144)}$$

$$C_4 = \frac{\begin{vmatrix} \phi_{10} & \phi_{11} & \phi_{12} & x \\ \phi_{20} & \phi_{21} & \phi_{22} & y \\ \phi_{30} & \phi_{31} & \phi_{32} & u \\ \phi_{40} & \phi_{41} & \phi_{42} & v \end{vmatrix}}{D} \qquad \text{(A-145)}$$

where

$$D = \begin{vmatrix} \phi_{10} & \phi_{11} & \phi_{12} & \phi_{14} \\ \phi_{20} & \phi_{21} & \phi_{22} & \phi_{24} \\ \phi_{30} & \phi_{31} & \phi_{32} & \phi_{34} \\ \phi_{40} & \phi_{41} & \phi_{42} & \phi_{44} \end{vmatrix} \qquad \text{(A-146)}$$

and

$$\phi_{10} = \frac{3}{4}\tau'^3 - 8\tau' + 8\sin\tau' \qquad \phi_{30} = 8(1-\cos\tau') - \frac{9}{4}\tau'^2$$

$$\phi_{11} = 8(1-\cos\tau') - 5\tau'\sin\tau' \qquad \phi_{31} = 5\tau'\cos\tau' - 3\sin\tau'$$

$$\phi_{12} = 5\tau'\cos\tau' - 11\sin\tau' + 6\tau' \qquad \phi_{32} = 5\tau'\sin\tau' - 6(1-\cos\tau')$$

$$\phi_{14} = 6(1-\cos\tau') - \frac{9}{4}\tau'^2 \qquad \phi_{34} = \frac{9}{2}\tau' - 6\sin\tau'$$

$$\phi_{20} = 4(1-\cos\tau') - \frac{3}{2}\tau'^2 \qquad \phi_{40} = 3\tau' - 4\sin\tau'$$

$$\phi_{21} = \frac{5}{2}\left[\tau'\cos\tau' - \sin\tau'\right] \qquad \phi_{41} = \frac{5}{2}\tau'\sin\tau'$$

$$\phi_{22} = \frac{5}{2}\tau'\sin\tau' - 4(1-\cos\tau') \qquad \phi_{42} = \frac{3}{2}\sin\tau' - \frac{5}{2}\tau'\cos\tau' \tag{A-147}$$

$$\phi_{24} = 3(\tau' - \sin\tau') \qquad \phi_{44} = -3(1-\cos\tau')$$

$$\frac{\partial A_z}{\partial z} = \frac{-2\sin^2\tau'}{\tau'^2 - \sin^2\tau'} \tag{A-148}$$

$$\frac{\partial A_z}{\partial w} = \frac{-(2\tau' - \sin 2\tau')}{\tau'^2 - \sin^2\tau'} \tag{A-149}$$

B. Lagrange Variables

1. Control Equations

$$A_R = \frac{\partial A_R}{\partial \Delta x_1}\Delta x_1 + \frac{\partial A_R}{\partial \Delta x_2}\Delta x_2 + \frac{\partial A_R}{\partial \Delta x_3}\Delta x_3 + \frac{\partial A_R}{\partial \Delta x_4}\Delta x_4 + \frac{\partial A_R}{\partial x_{30}} x_{30} \tag{A-150}$$

$$A_S = \frac{\partial A_S}{\partial \Delta x_1}\Delta x_1 + \frac{\partial A_S}{\partial \Delta x_2}\Delta x_2 + \frac{\partial A_S}{\partial \Delta x_3}\Delta x_3 + \frac{\partial A_S}{\partial \Delta x_4}\Delta x_4 + \frac{\partial A_S}{\partial x_{30}} x_{30} \tag{A-151}$$

$$A_W = \frac{\partial A_W}{\partial \Delta x_5}\Delta x_5 + \frac{\partial A_W}{\partial \Delta x_6}\Delta x_6 \tag{A-152}$$

2. Guidance Coefficients-Transfer

$$\frac{\partial A_R}{\partial \Delta x_1} = \frac{-4\sin\tau'\sin\frac{\tau'}{2}}{\tau'(5\tau' + 3\sin\tau') - 16(1-\cos\tau')} \tag{A-153}$$

$$\frac{\partial A_R}{\partial \Delta x_2} = \frac{2\cos\tau'}{5\tau' - 3\sin\tau'} \tag{A-154}$$

$$\frac{\partial A_R}{\partial \Delta x_3} = \frac{2\tau'\sin\tau'}{\tau'(5\tau' + 3\sin\tau') - 16(1 - \cos\tau')} \tag{A-155}$$

$$\frac{\partial A_S}{\partial \Delta x_1} = \frac{8\cos\tau'\sin\frac{\tau'}{2} - \frac{1}{2}(5\tau' + 3\sin\tau')}{\tau'(5\tau' + 3\sin\tau') - 16(1 - \cos\tau')} \tag{A-156}$$

$$\frac{\partial A_S}{\partial \Delta x_2} = \frac{4\sin\tau'}{5\tau' - 3\sin\tau'} \tag{A-157}$$

$$\frac{\partial A_W}{\partial \Delta x_5} = \frac{\cos\tau'\sin^2\tau'}{\tau'^2 - \sin^2\tau'} \tag{A-158}$$

$$\frac{\partial A_W}{\partial \Delta x_6} = \frac{-\frac{1}{2}\cos\tau'(2\tau' - \sin 2\tau')}{\tau'^2 - \sin^2\tau'} \tag{A-159}$$

$$\frac{\partial A_S}{\partial \Delta x_3} = \frac{4(2\sin\frac{\tau'}{2} - \tau'\cos\tau')}{\tau'(5\tau' + 3\sin\tau') - 16(1 - \cos\tau')} \tag{A-160}$$

3. Guidance Coefficients-Rendezvous

$$\frac{\partial A_R}{\partial \Delta x_1} = \frac{4\sin\tau'\sin\frac{\tau'}{2}}{Q} - \frac{\frac{3}{8}\tau\left[5\tau' - 3\sin\tau' + 2\cos\tau'(3\tau'\cos\frac{\tau'}{2} - 8\sin\frac{\tau'}{2})\right]}{B} \tag{A-161}$$

$$\frac{\partial A_R}{\partial \Delta x_2} = \frac{2\tau'\cos\tau'(\frac{3}{16}\tau^2 + 1) + \cos\frac{\tau'}{2}(5\tau' - 3\sin\tau') + 2(3\tau'\cos\frac{\tau'}{2} - 8\sin\frac{\tau'}{2})(1 + \cos\tau'\cos\frac{\tau'}{2})}{B} \tag{A-162}$$

$$\frac{\partial A_R}{\partial \Delta x_3} = \frac{-2\tau' \sin\tau'}{Q} + \frac{\sin\frac{\tau'}{2}\left[5\tau' - 3\sin\tau' + 2\cos\tau'\left(3\tau'\cos\frac{\tau'}{2} - 8\sin\frac{\tau'}{2}\right)\right]}{B} \tag{A-163}$$

$$\frac{\partial A_R}{\partial \Delta x_4} = -\frac{\frac{1}{2}\left[5\tau' - 3\sin\tau' + 2\cos\tau'\left(3\tau'\cos\frac{\tau'}{2} - 8\sin\frac{\tau'}{2}\right)\right]}{B} \tag{A-164}$$

$$\frac{\partial A_R}{\partial x_{30}} = \frac{2\sin\frac{\tau'}{2}\left[5\tau' - 3\sin\tau' + 2\cos\tau'\left(3\tau'\cos\frac{\tau'}{2} - 8\sin\frac{\tau'}{2}\right)\right]}{B} \tag{A-165}$$

$$\frac{\partial A_S}{\partial \Delta x_1} = \frac{\frac{1}{2}\left(5\tau' + 3\sin\tau' - 16\sin\frac{\tau'}{2}\cos\tau'\right)}{Q} - \frac{\frac{3}{16}\tau'\left[3\tau'(5\tau' - 3\sin\tau') + 8\sin\tau'\left(3\tau'\cos\frac{\tau'}{2} - 8\sin\frac{\tau'}{2}\right)\right]}{B} \tag{A-166}$$

$$\frac{\partial A_S}{\partial \Delta x_2} = \frac{4\tau'\sin\tau'\left(\frac{3}{16}\tau'^2 + 1\right) + \frac{3}{2}\tau'\cos\frac{\tau'}{2}(5\tau' - 3\sin\tau')}{B} + \frac{\left(3\tau'\cos\frac{\tau'}{2} - 8\sin\frac{\tau'}{2}\right)\left(3\tau' + 4\sin\tau'\cos\frac{\tau'}{2}\right)}{B} \tag{A-167}$$

$$\frac{\partial A_S}{\partial \Delta x_3} = \frac{4\left(\tau'\cos\tau' - 2\sin\frac{\tau'}{2}\right)}{Q} + \frac{\frac{1}{2}\sin\frac{\tau'}{2}\left[3\tau'(5\tau' - 3\sin\tau') + 8\sin\tau'\left(3\tau'\cos\frac{\tau'}{2} - 8\sin\frac{\tau'}{2}\right)\right]}{B} \tag{A-168}$$

$$\frac{\partial A_S}{\partial \Delta x_4} = -\frac{\frac{1}{4}\left[3\tau'(5\tau' - 3\sin\tau') + 8\sin\tau'\left(3\tau'\cos\frac{\tau'}{2} - 8\sin\frac{\tau'}{2}\right)\right]}{B} \tag{A-169}$$

$$\frac{\partial A_S}{\partial x_{30}} = \frac{\sin\frac{\tau'}{2}\left[3\tau'(5\tau' - 3\sin\tau') + 8\sin\tau'\left(3\tau'\cos\frac{\tau'}{2} - 8\sin\frac{\tau'}{2}\right)\right]}{B} \tag{A-170}$$

$$\frac{\partial A_W}{\partial \Delta x_5} = \frac{2\sin\tau'(\tau' + \sin 2\tau')}{\tau'^2 - \sin^2\tau'} \tag{A-171}$$

$$\frac{\partial A_W}{\partial \Delta x_6} = -\frac{2\left[\sin\tau' + \cos\tau'(\tau' - \sin 2\tau')\right]}{\tau'^2 - \sin^2\tau'} \tag{A-172}$$

where

$$Q = 16(1-\cos\tau') - \tau'(5\tau' + 3\sin\tau') \tag{A-173}$$

$$B = \tau'(5\tau' - 3\sin\tau')\left(\frac{3}{16}\tau'^2 + 1\right) - 2\left(8\sin\frac{\tau'}{2} - 3\tau'\cos\frac{\tau'}{2}\right)^2 \tag{A-174}$$

FIG. I

RECTANGULAR COORDINATE SYSTEM

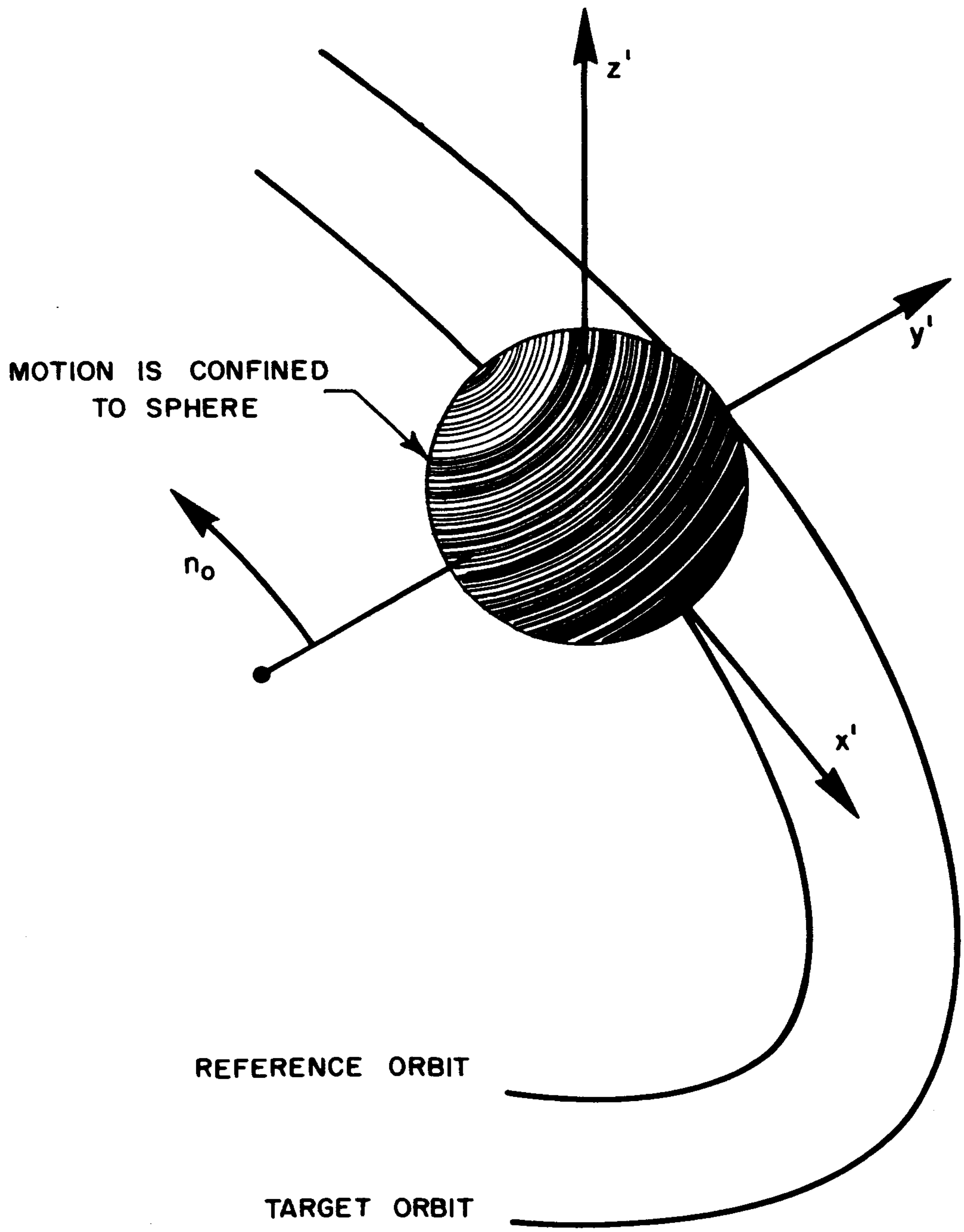


FIG. 2

SPHERICAL COORDINATE SYSTEM

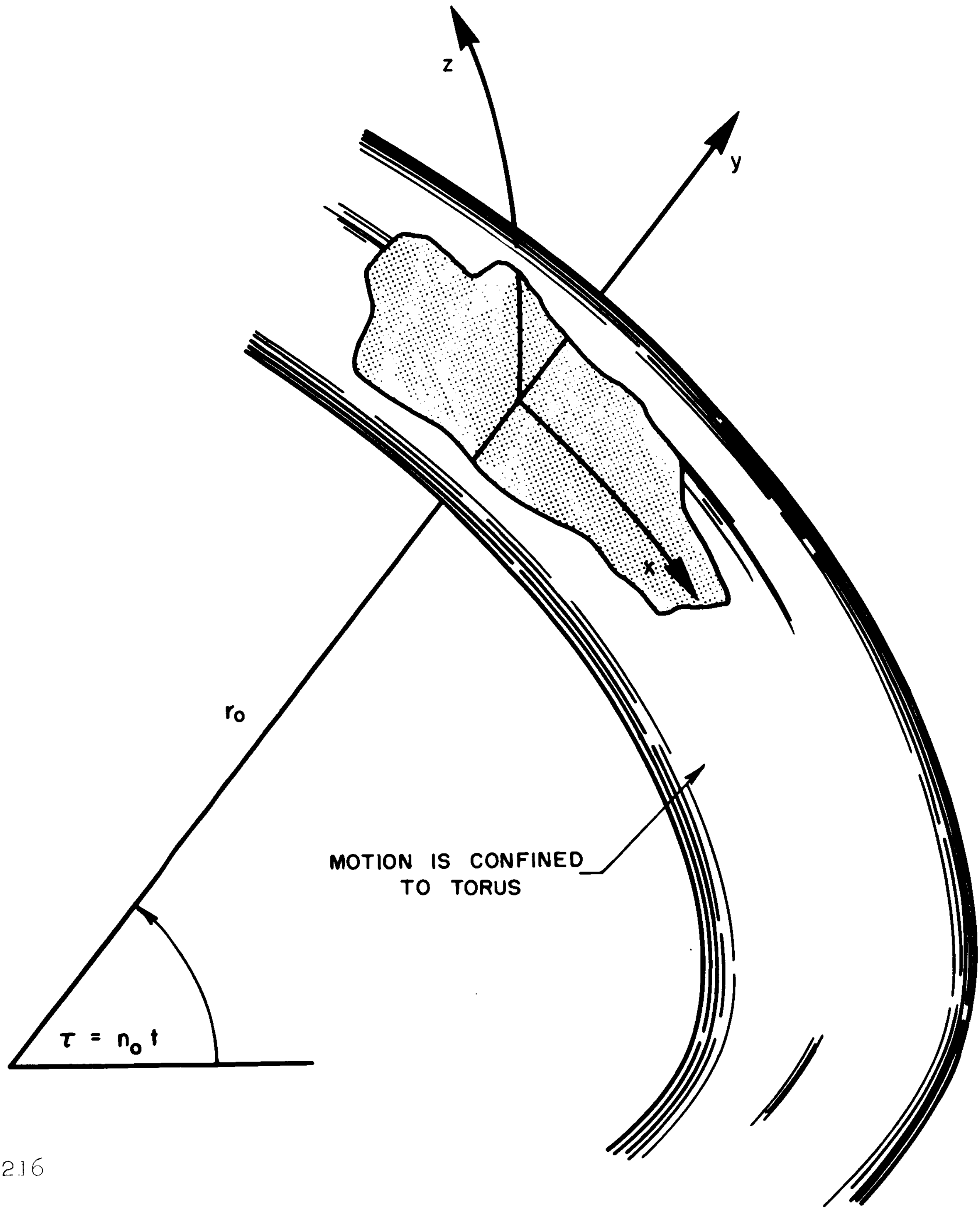
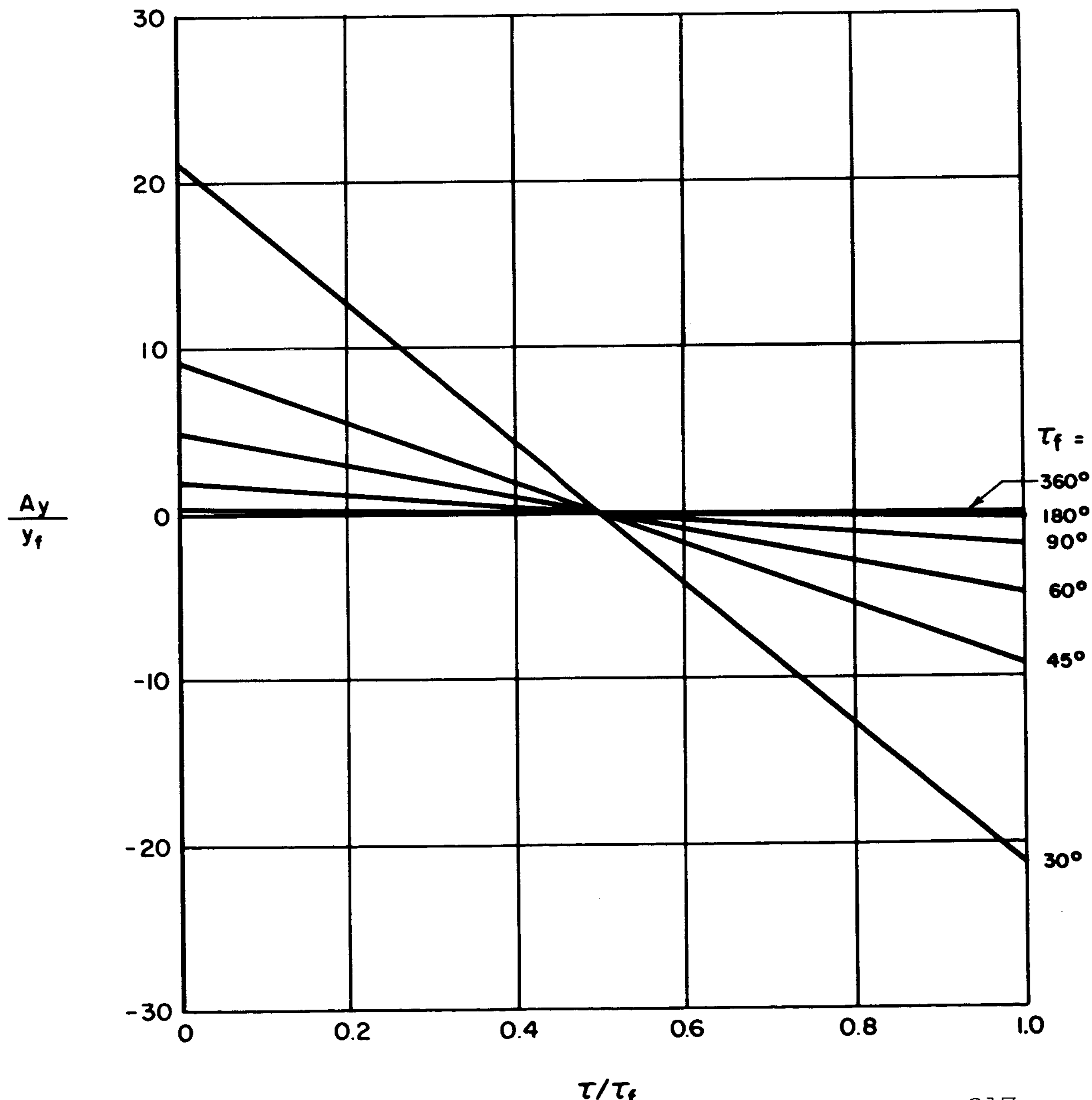


FIG. 3

RADIAL ACCELERATION

CIRCLE-TO-CIRCLE TRANSFER



CIRCUMFERENTIAL ACCELERATION
CIRCLE – TO – CIRCLE TRANSFER

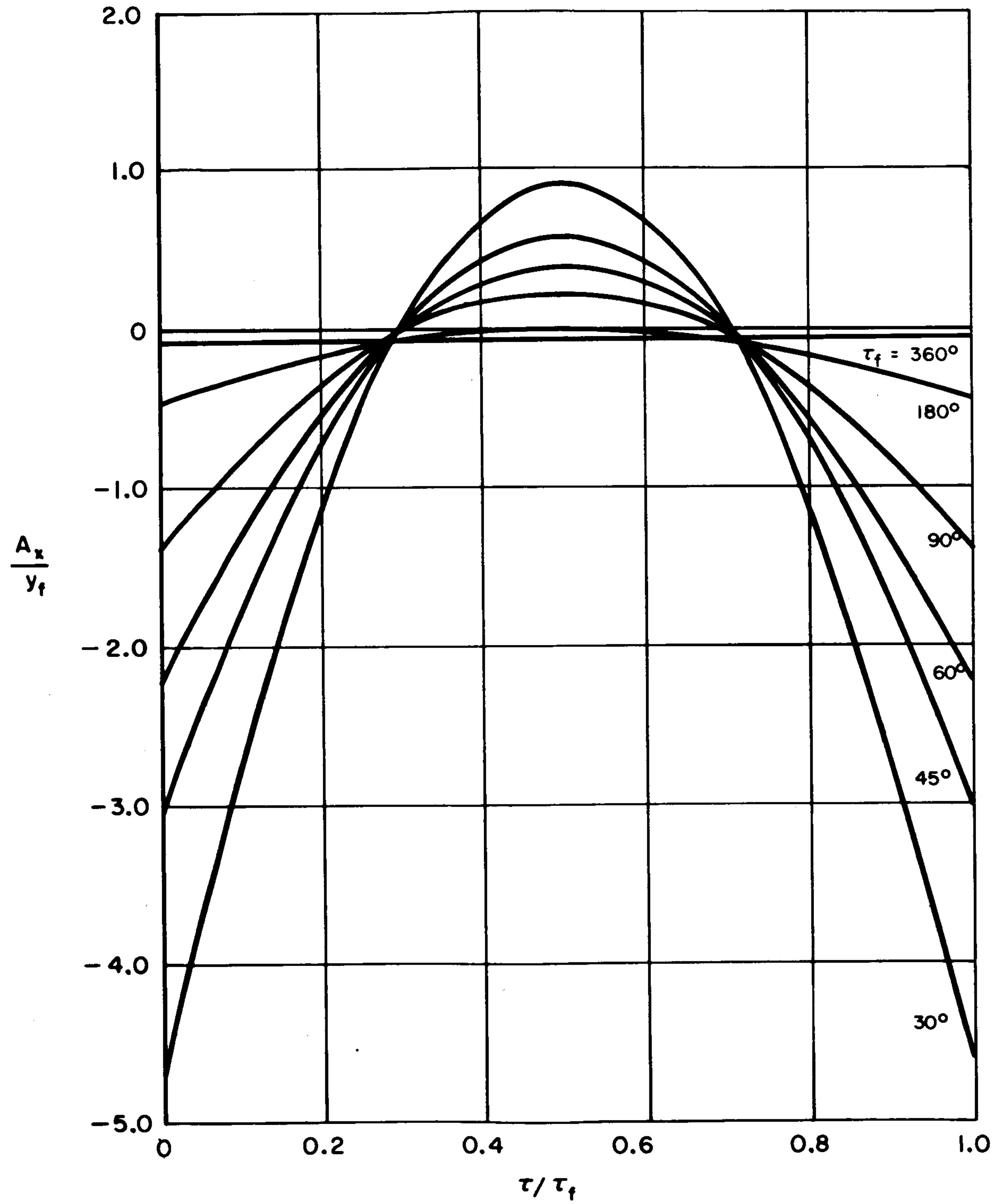


FIG. 5

NORMAL ACCELERATION

CIRCLE-TO-CIRCLE TRANSFER

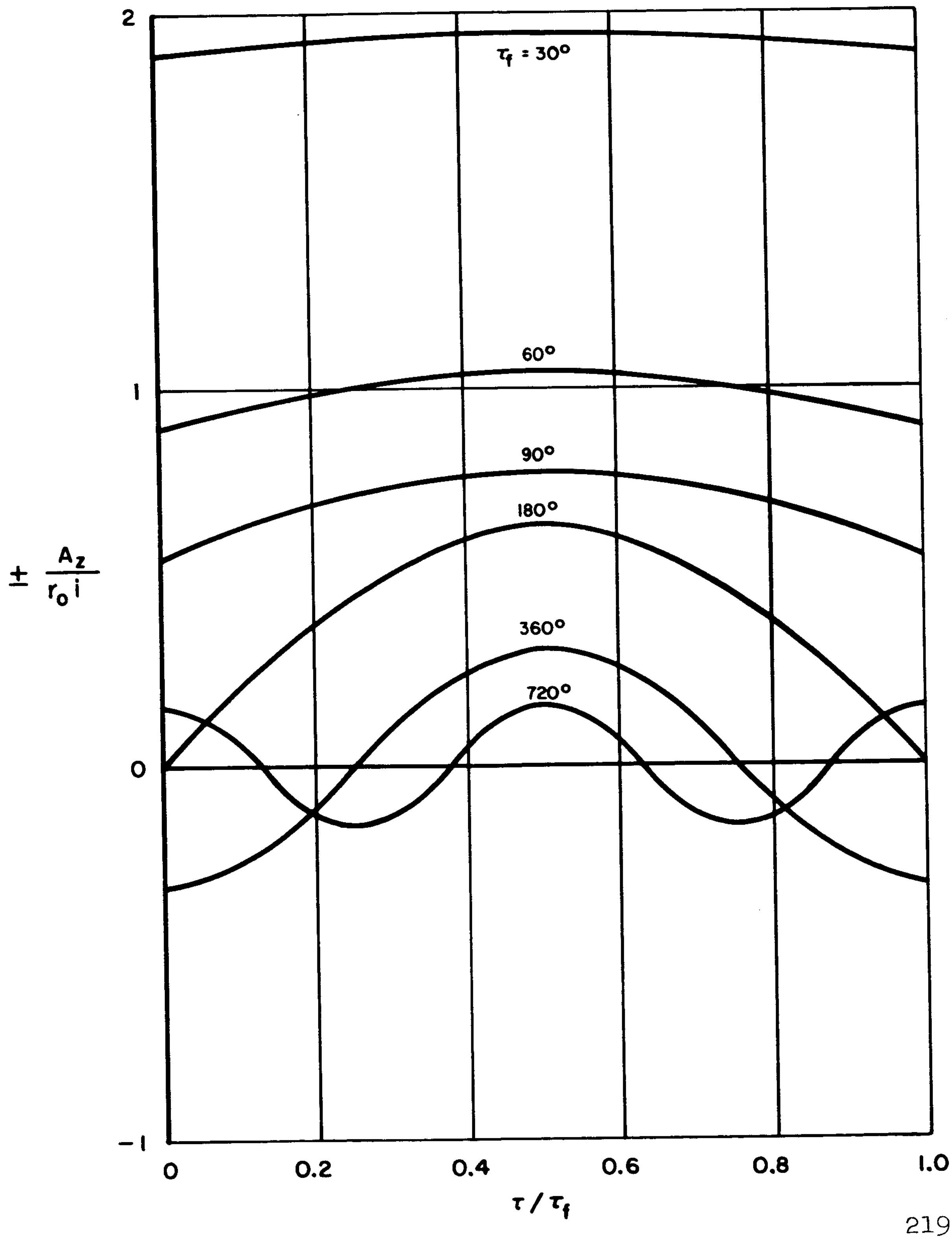
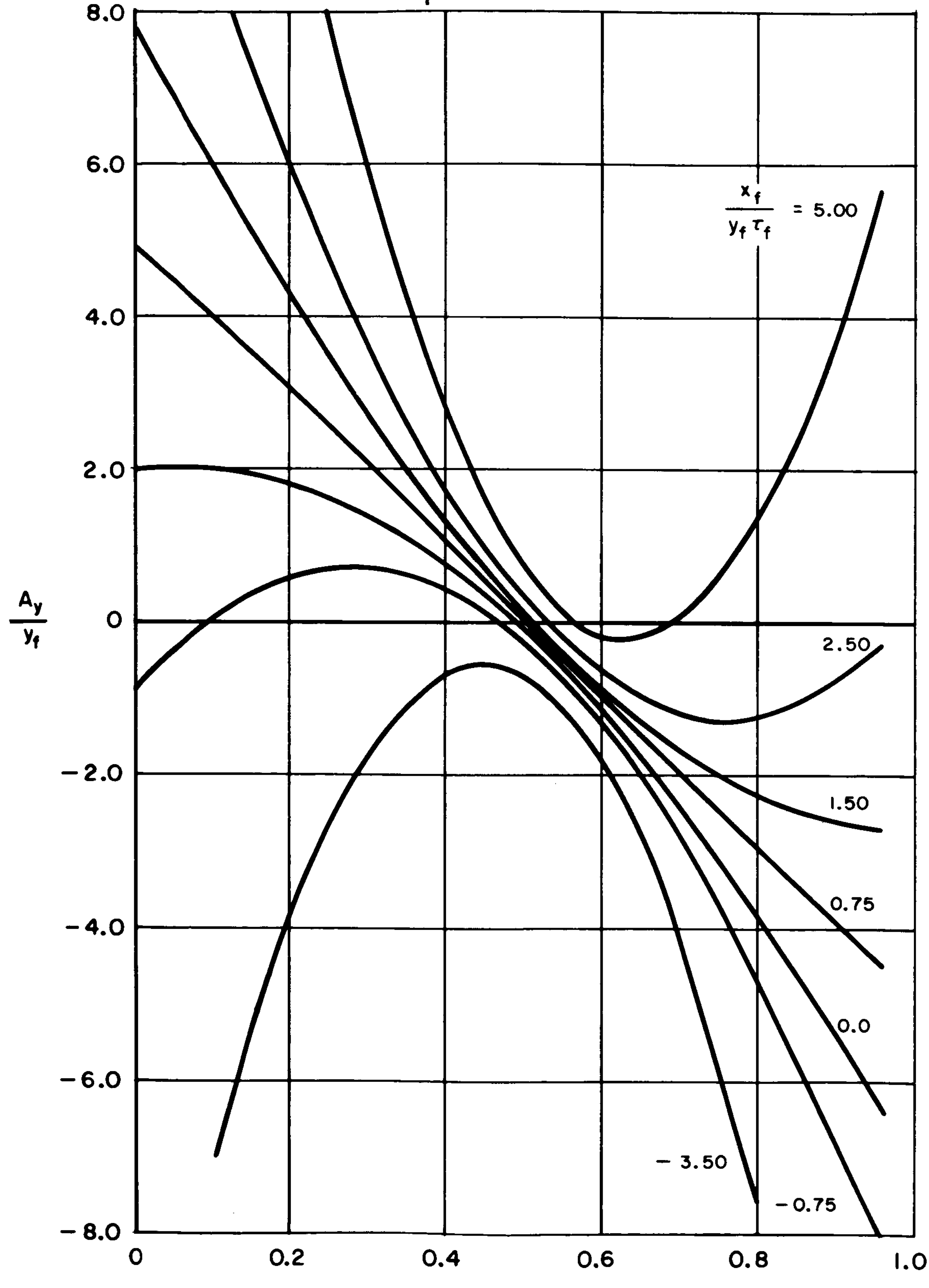


FIG. 6

RADIAL ACCELERATION

CIRCLE - TO - CIRCLE RENDEZVOUS

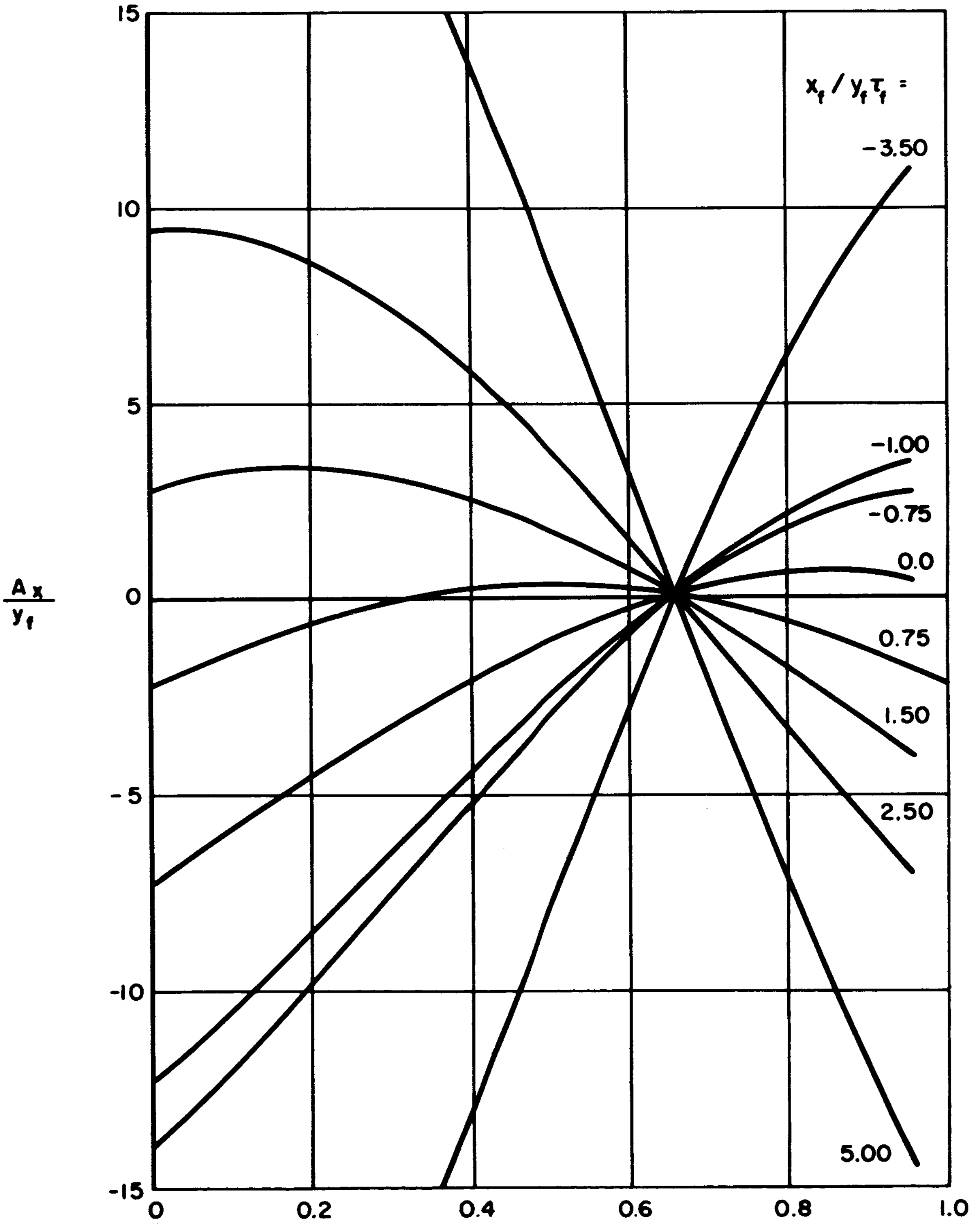
$\tau_f = 60°$



CIRCUMFERENTIAL ACCELERATION

CIRCLE - TO - CIRCLE RENDEZVOUS

$\tau_f = 60°$

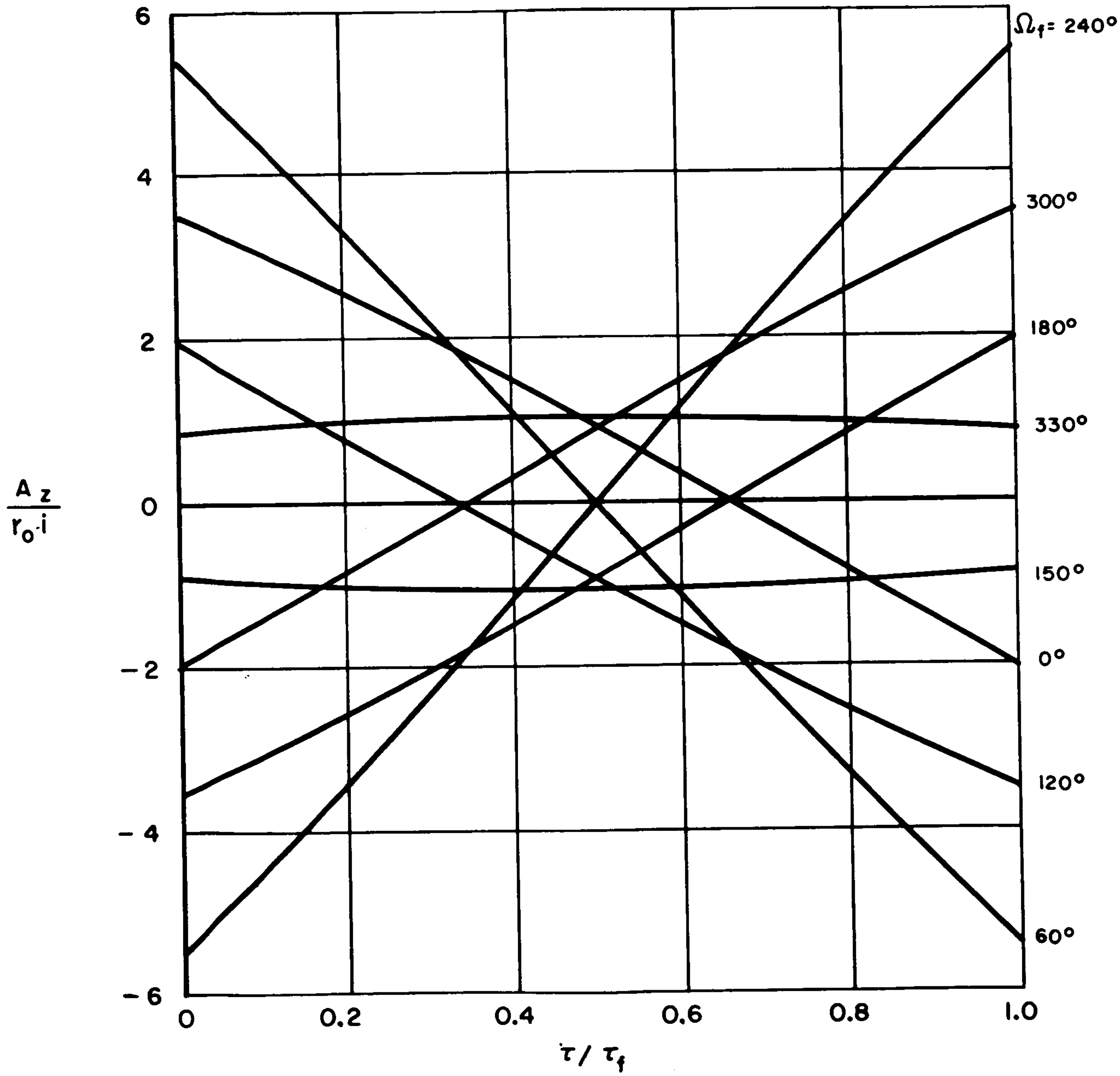


NORMAL ACCELERATION

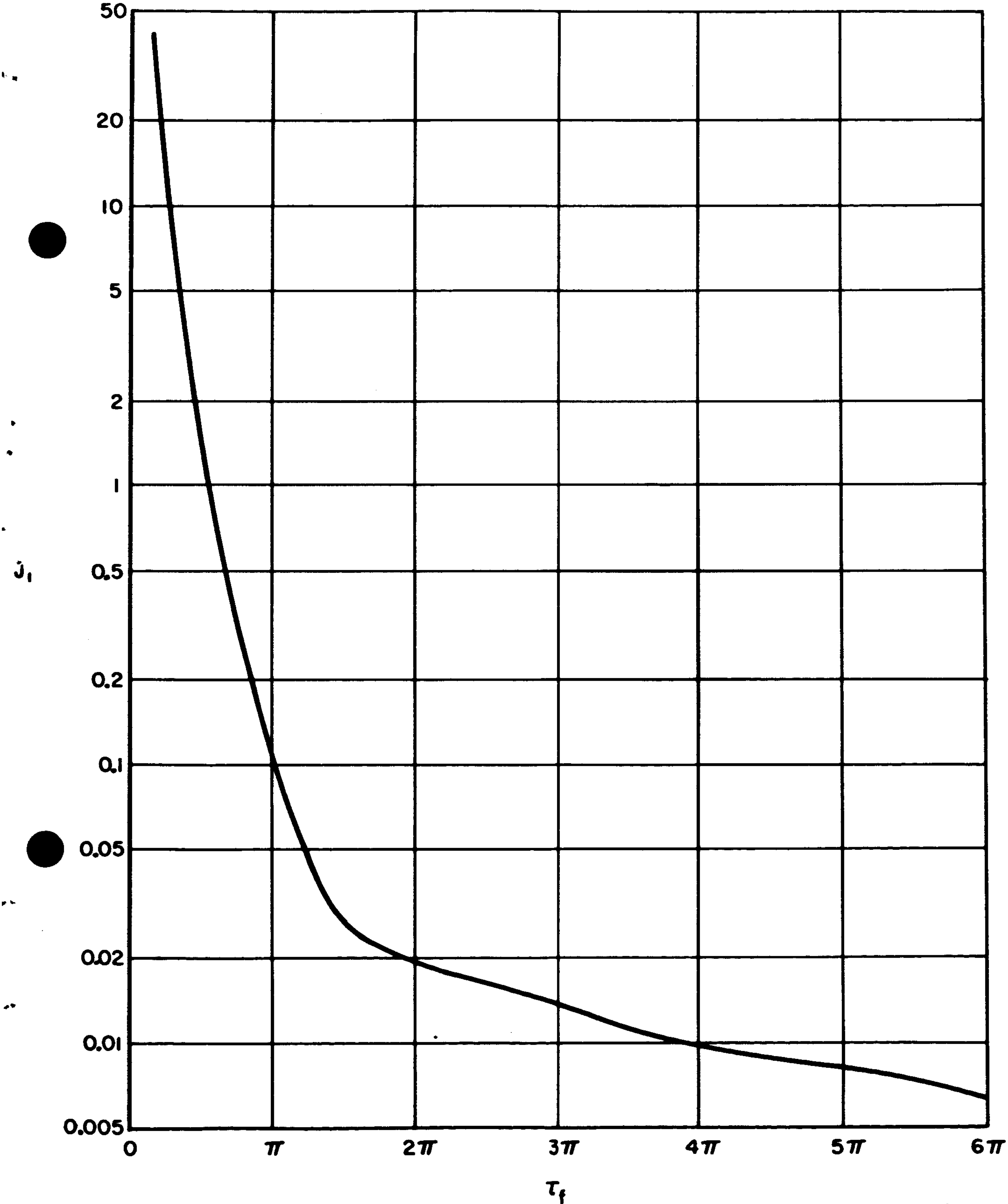
CIRCLE - TO - CIRCLE RENDEZVOUS

$\tau_f = 60°$

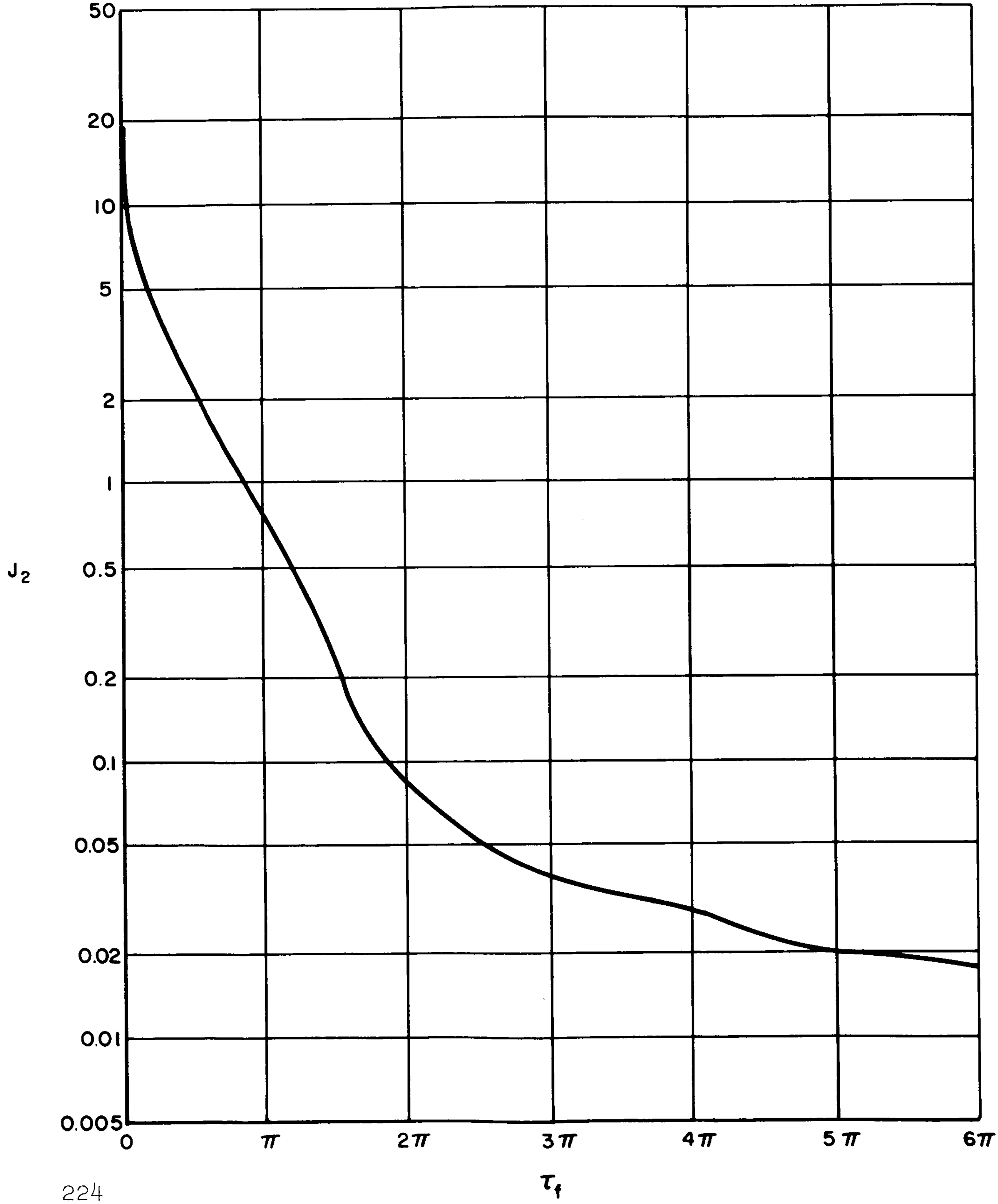
OPTIMUM TRANSFERS : $\Omega_f = 150°, 330°$



IN-PLANE COMPONENT OF J
CIRCLE-TO-CIRCLE TRANSFER



IN-PLANE COMPONENT OF J
CIRCLE-TO-CIRCLE RENDEZVOUS



OUT-OF-PLANE COMPONENT OF J

CIRCLE-TO-CIRCLE RENDEZVOUS

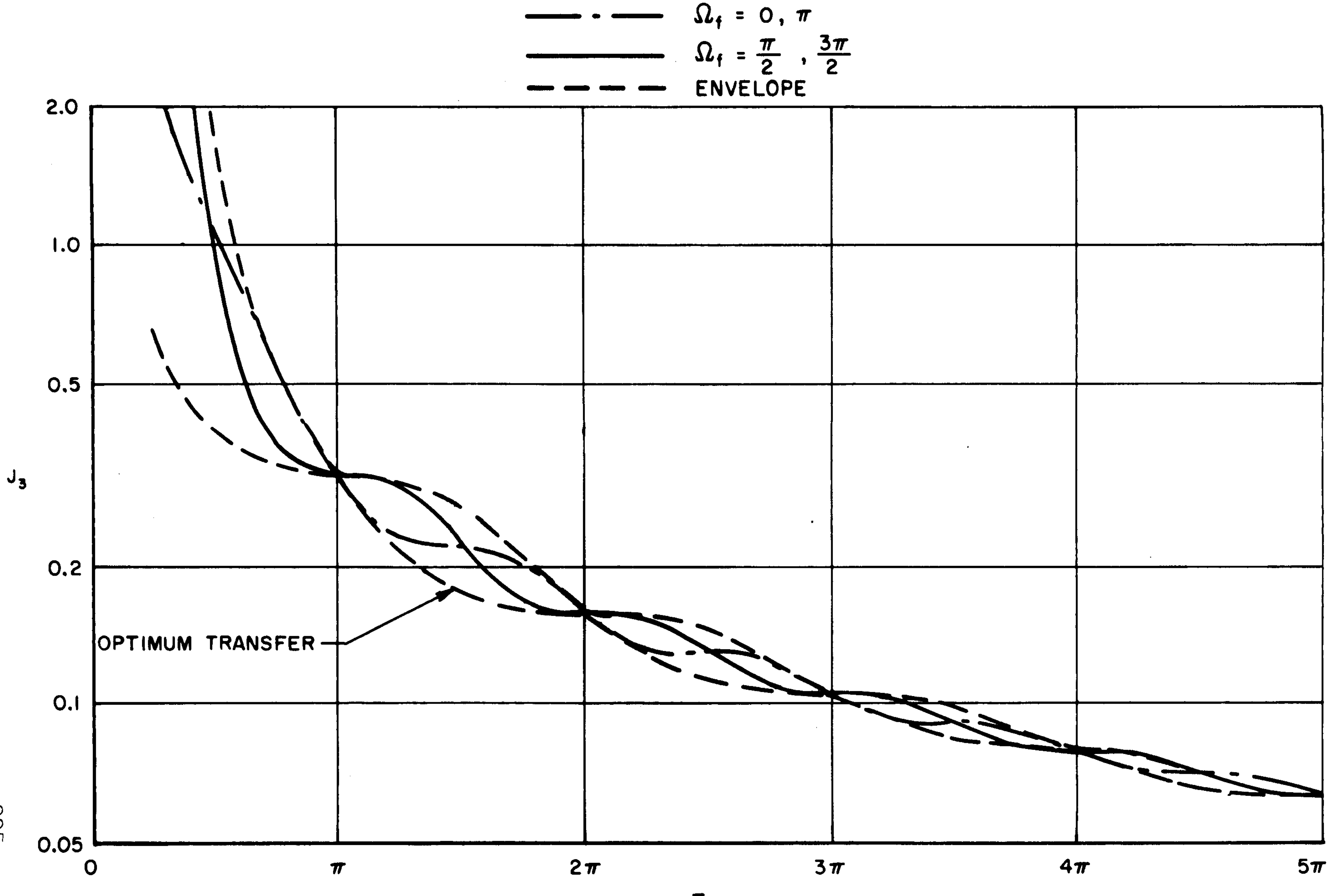


FIG. 11

OPTIMUM EARTH - VENUS TRANSFER

UNINCLINED CIRCULAR TERMINAL ORBITS

⊙ — LINEARIZED ANALYSIS

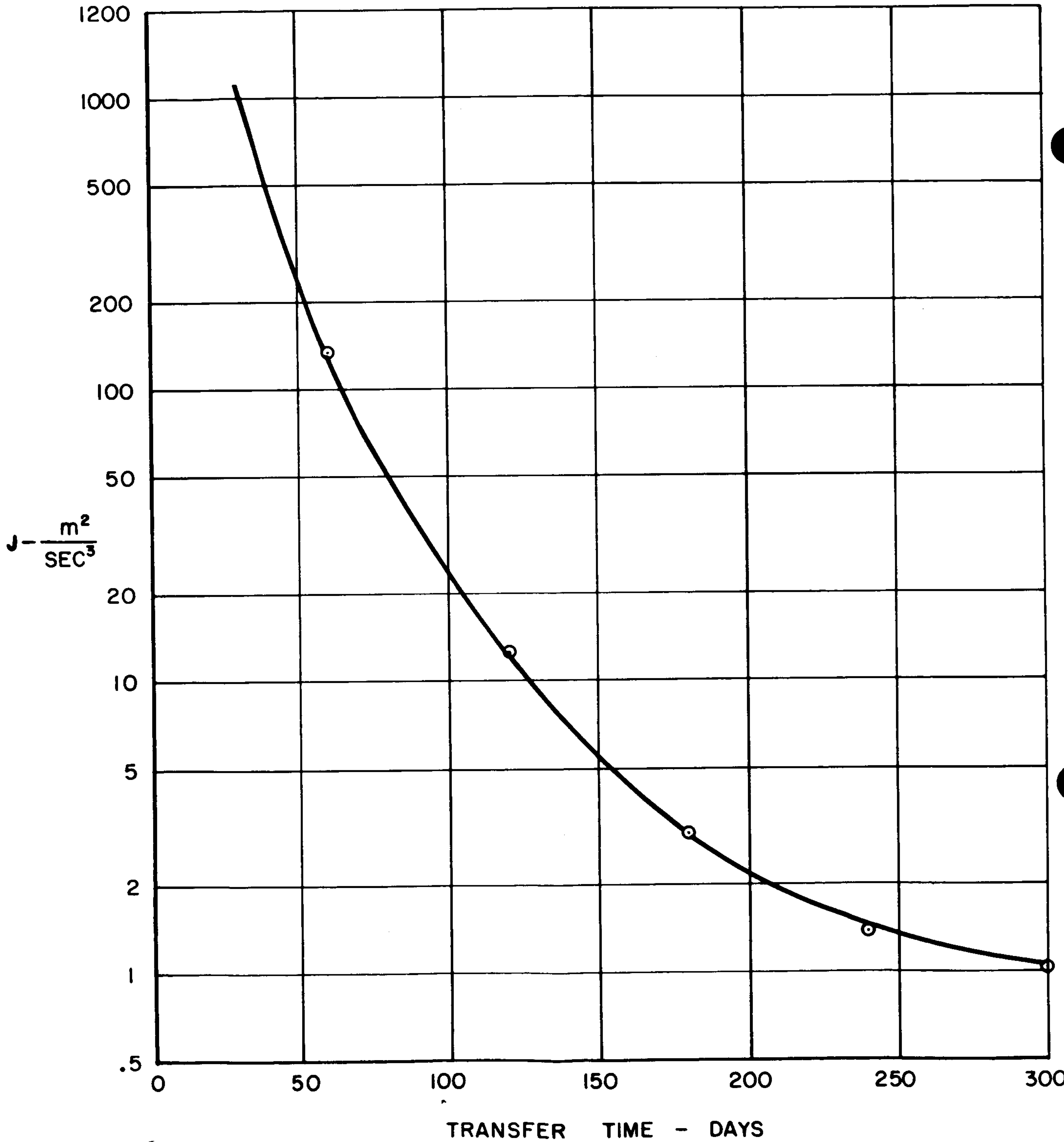
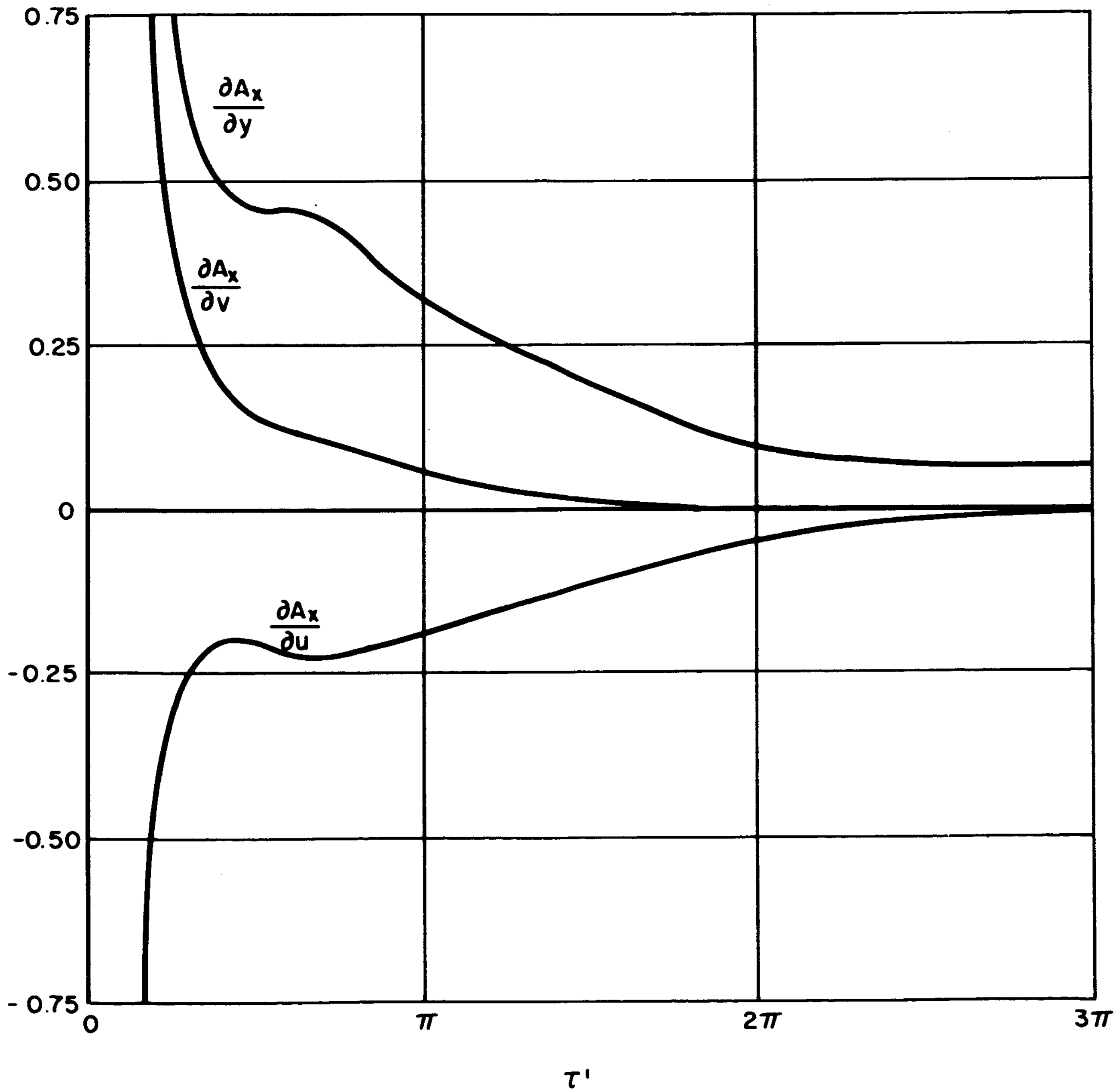


FIG. 13

GUIDANCE COEFFICIENTS FOR OPTIMUM CONTROL
ORBIT TRANSFER

$$A_x = \frac{\partial A_x}{\partial y} y + \frac{\partial A_x}{\partial u} u + \frac{\partial A_x}{\partial v} v$$



GUIDANCE COEFFICIENTS FOR OPTIMUM CONTROL

ORBIT TRANSFER

$$A_y = \frac{\partial A_y}{\partial y} y + \frac{\partial A_y}{\partial u} u + \frac{\partial A_y}{\partial v} v$$

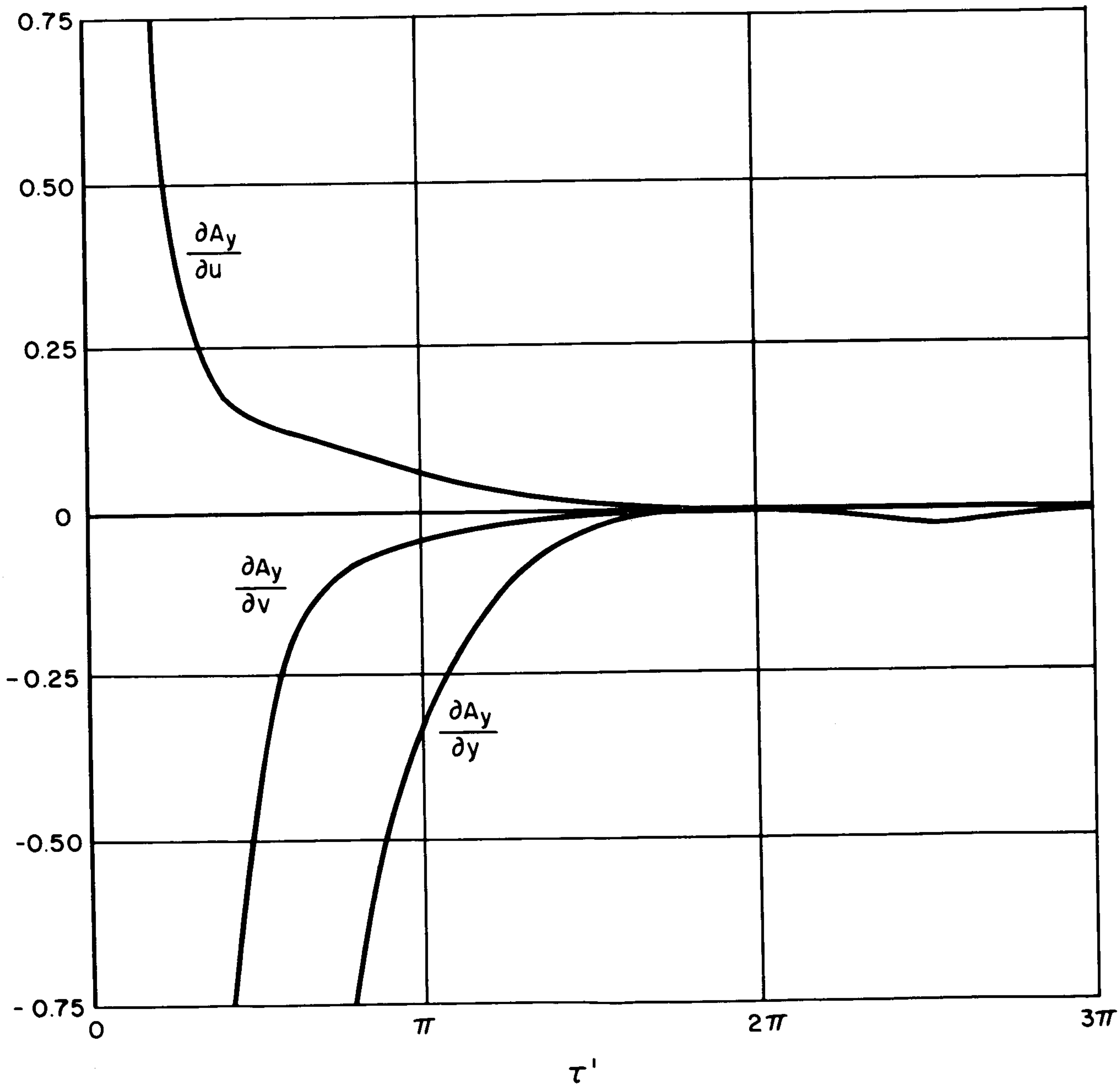
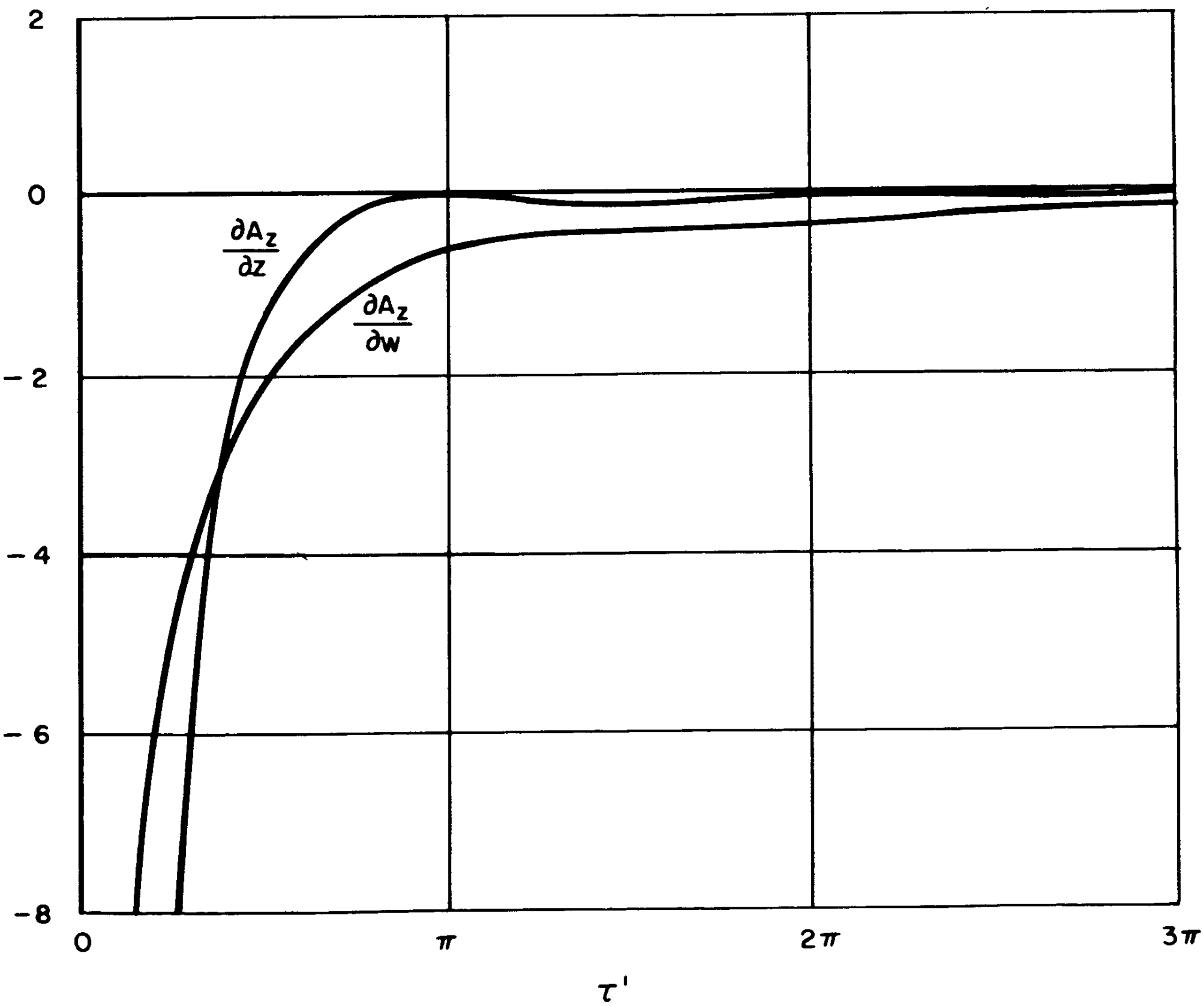


FIG. 15

GUIDANCE COEFFICIENTS FOR OPTIMUM CONTROL

ORBIT TRANSFER

$$A_z = \frac{\partial A_z}{\partial w} w + \frac{\partial A_z}{\partial z} z$$



GUIDANCE COEFFICIENTS FOR OPTIMUM CONTROL
ORBIT TRANSFER

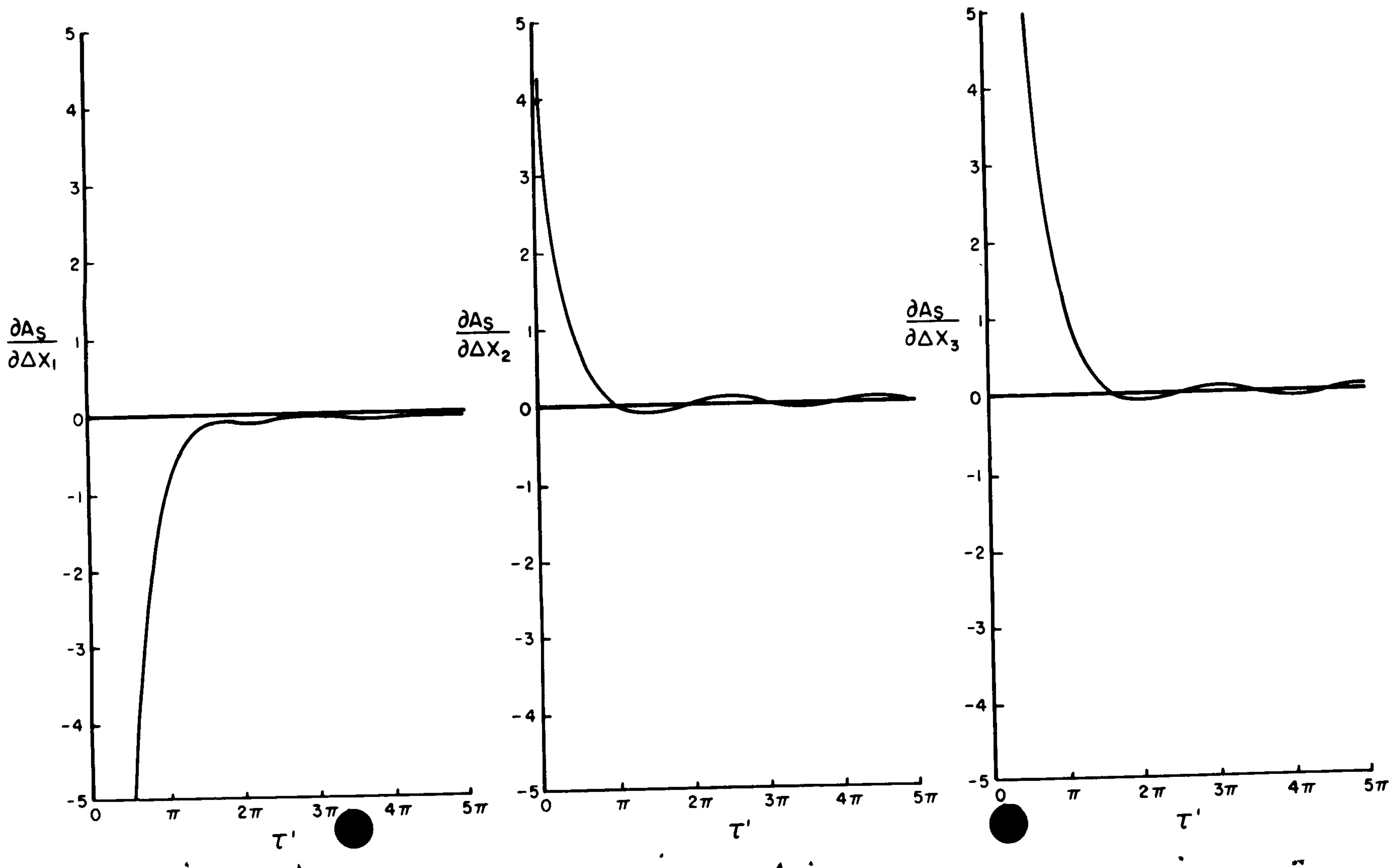


FIG.16

GUIDANCE COEFFICIENTS FOR OPTIMUM CONTROL
ORBIT TRANSFER

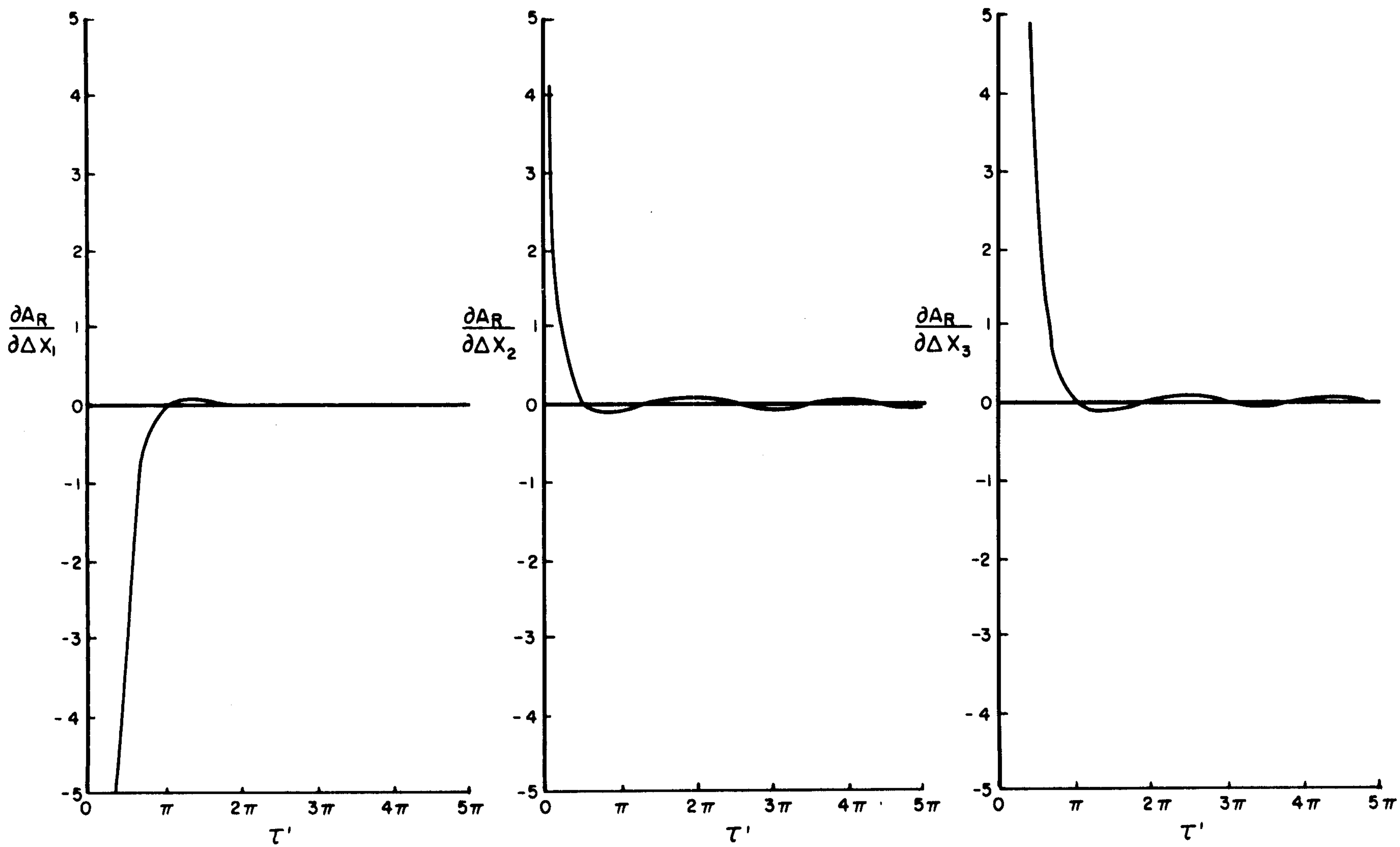
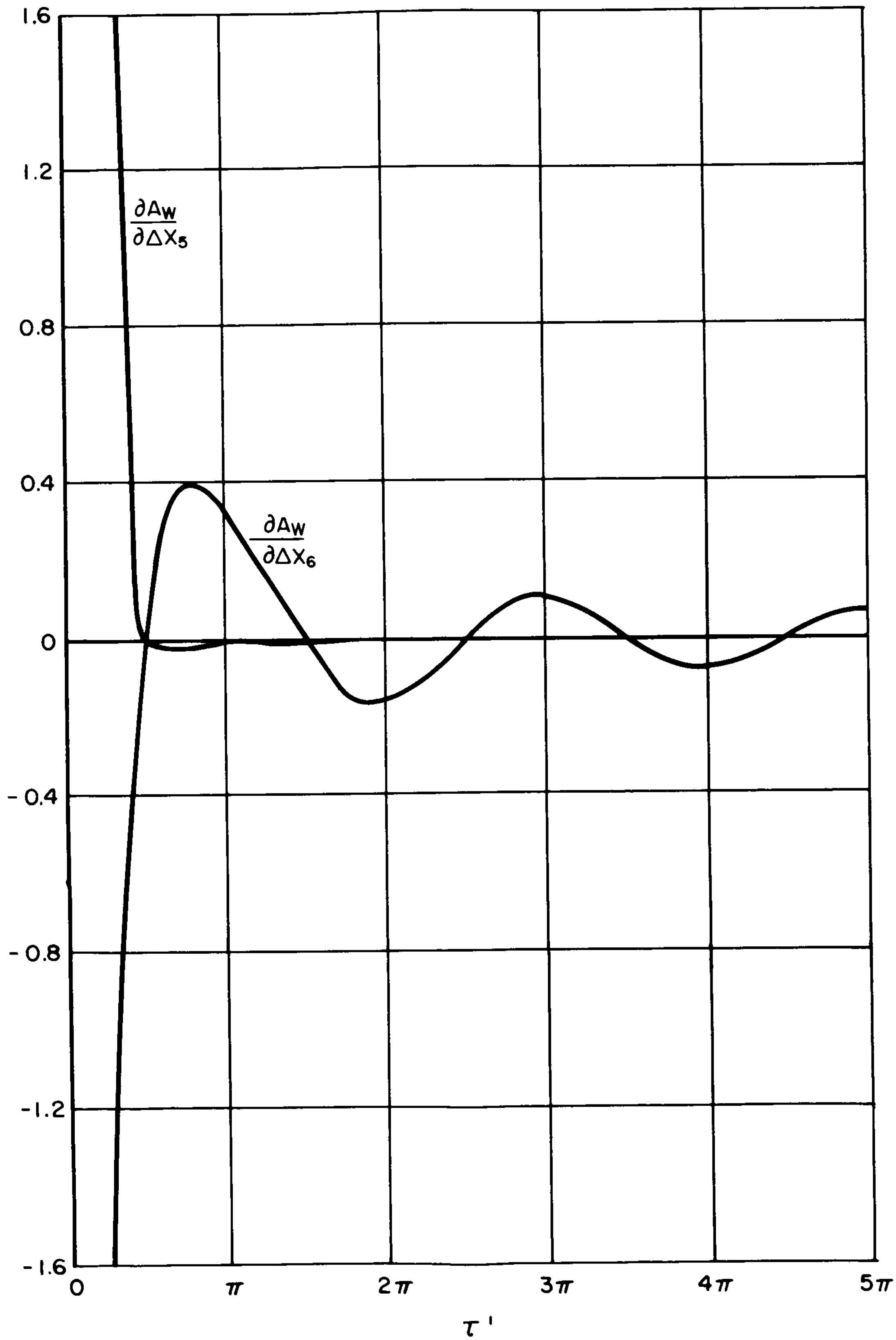


FIG. 17

GUIDANCE COEFFICIENTS FOR OPTIMUM CONTROL

ORBIT TRANSFER



GUIDANCE COEFFICIENTS FOR OPTIMUM CONTROL
ORBITAL RENDEZVOUS

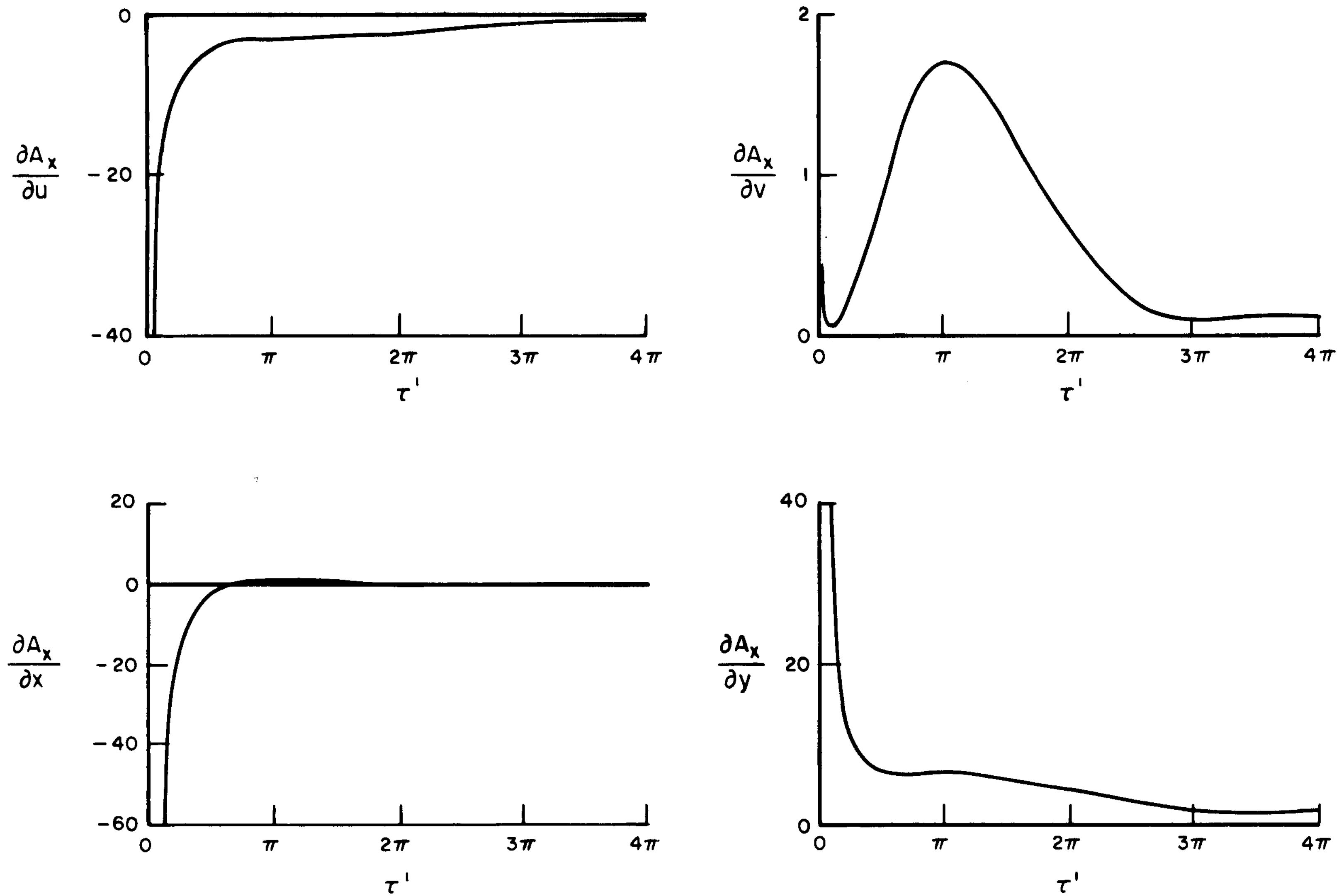


FIG. 19

GUIDANCE COEFFICIENTS FOR OPTIMUM CONTROL
ORBITAL RENDEZVOUS

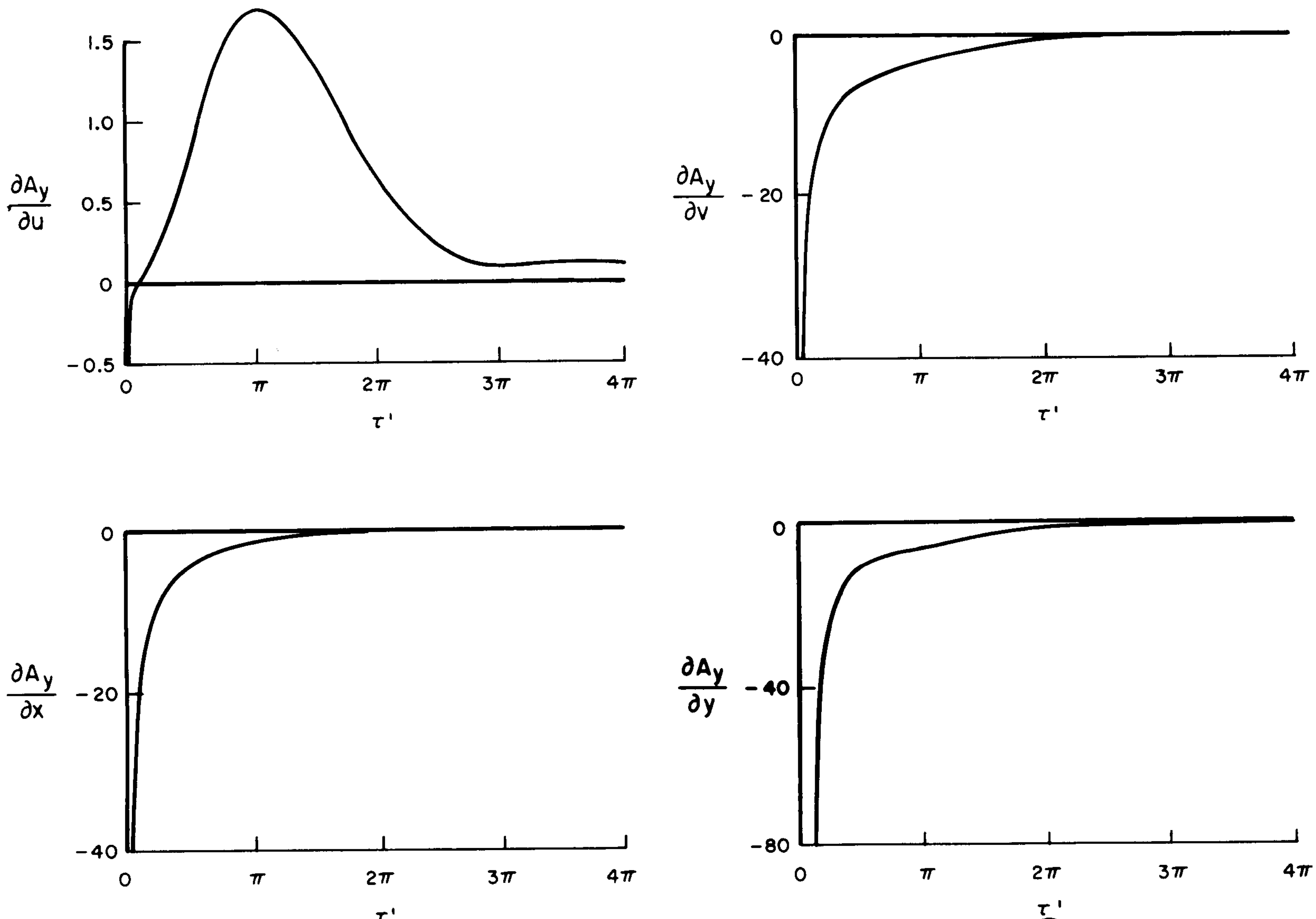
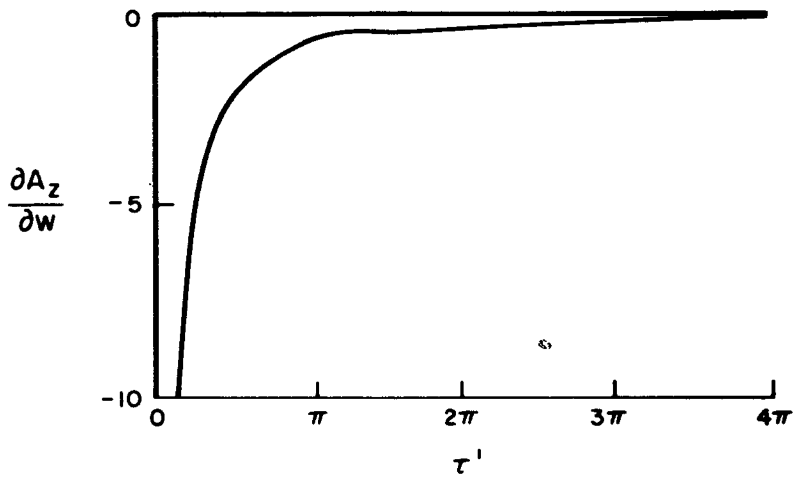
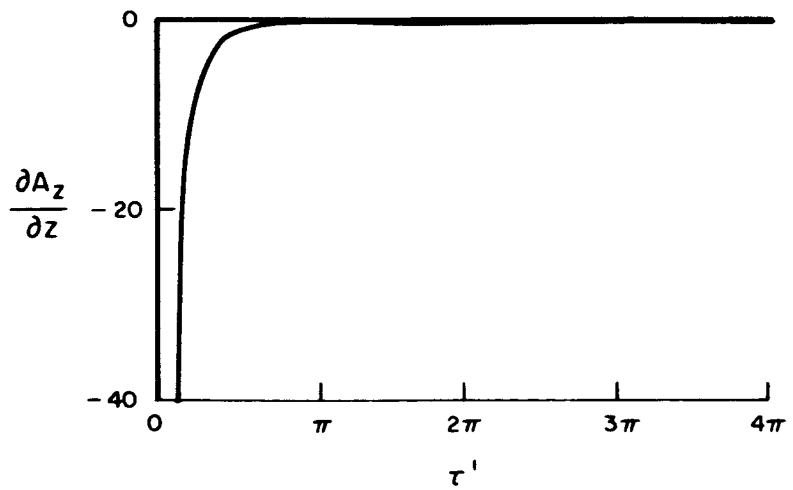


FIG. 20

GUIDANCE COEFFICIENTS FOR OPTIMUM CONTROL

ORBITAL RENDEZVOUS





GUIDANCE COEFFICIENTS FOR OPTIMUM CONTROL

ORBITAL RENDEZVOUS

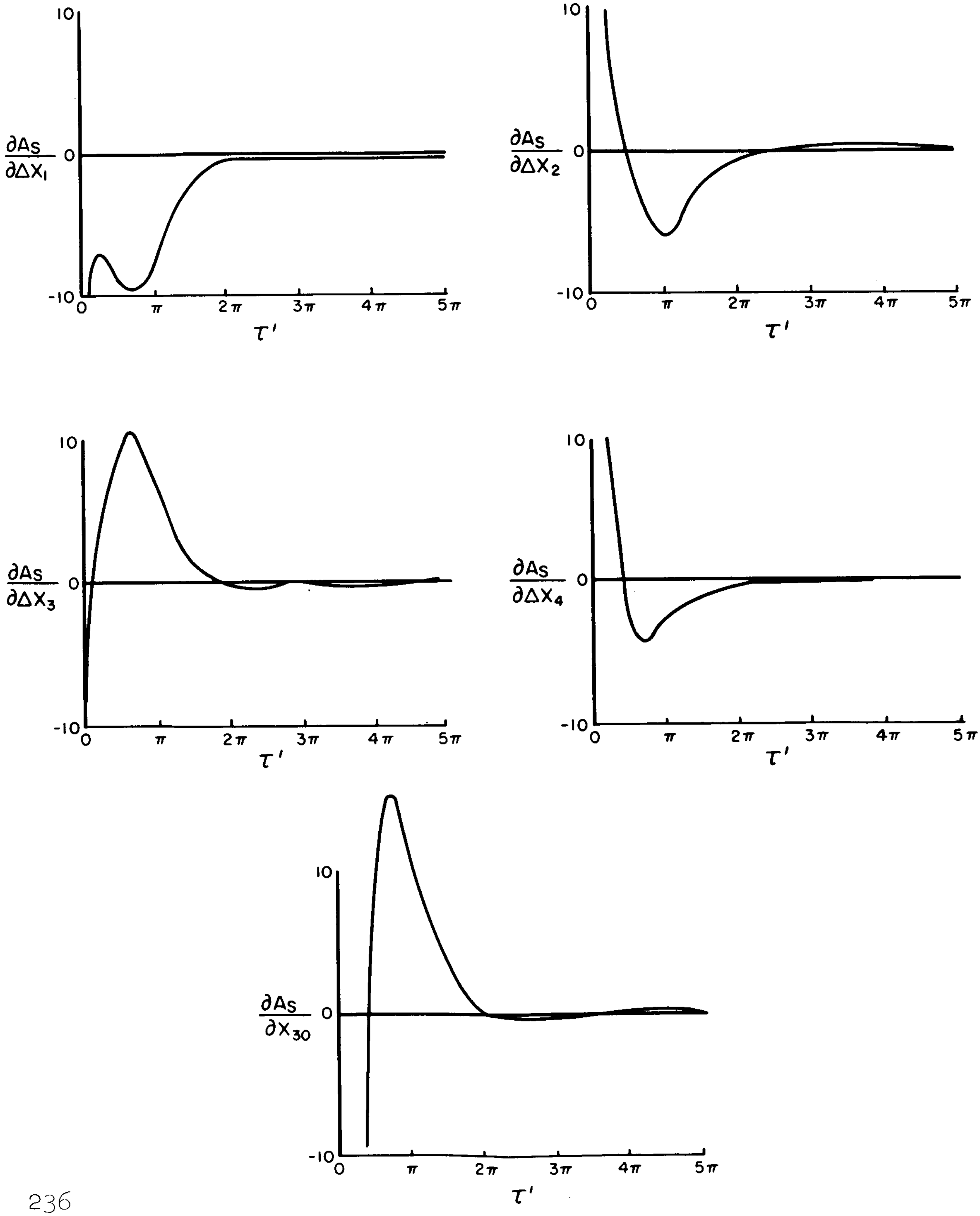
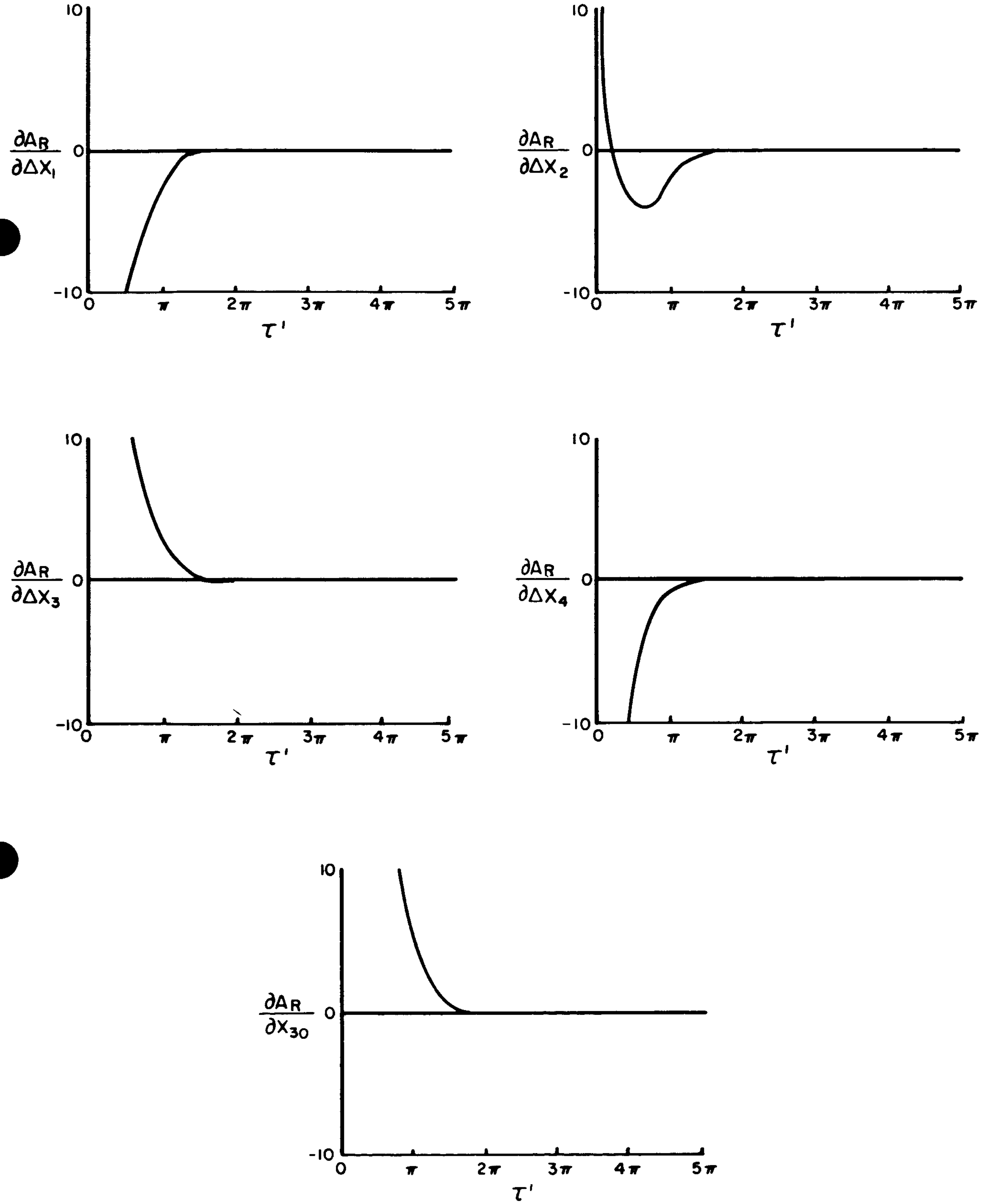


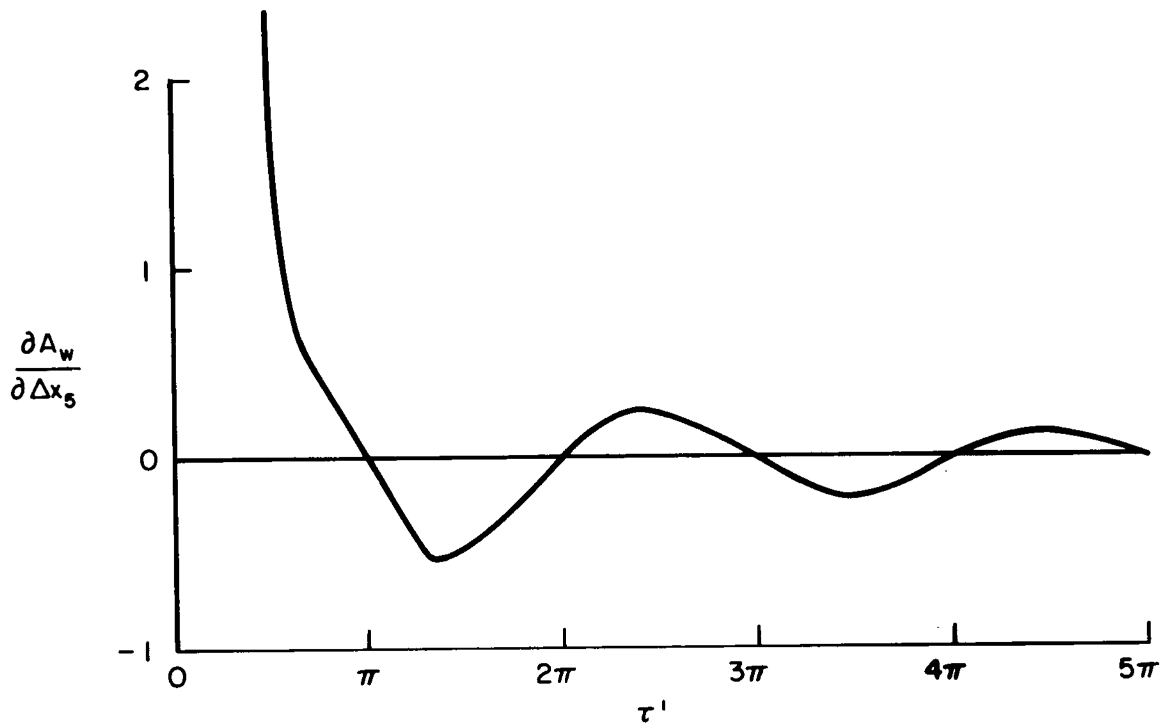
FIG. 23

GUIDANCE COEFFICIENTS FOR OPTIMUM CONTROL

ORBITAL RENDEZVOUS



GUIDANCE COEFFICIENTS FOR OPTIMUM CONTROL ORBITAL RENDEZVOUS



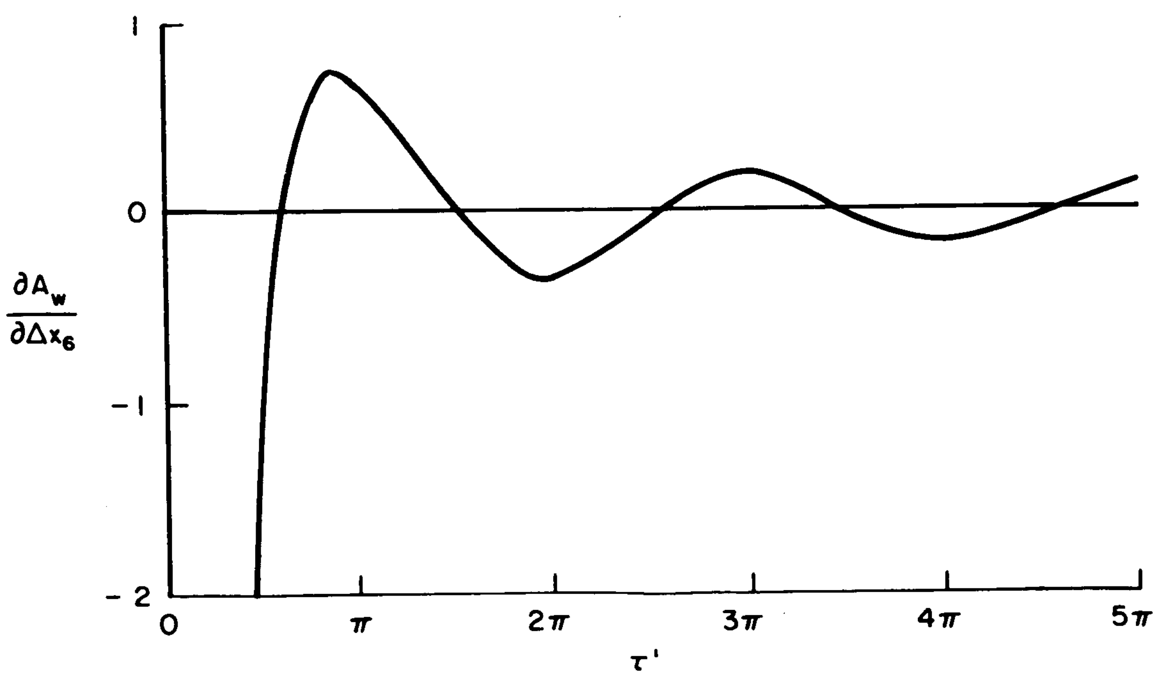
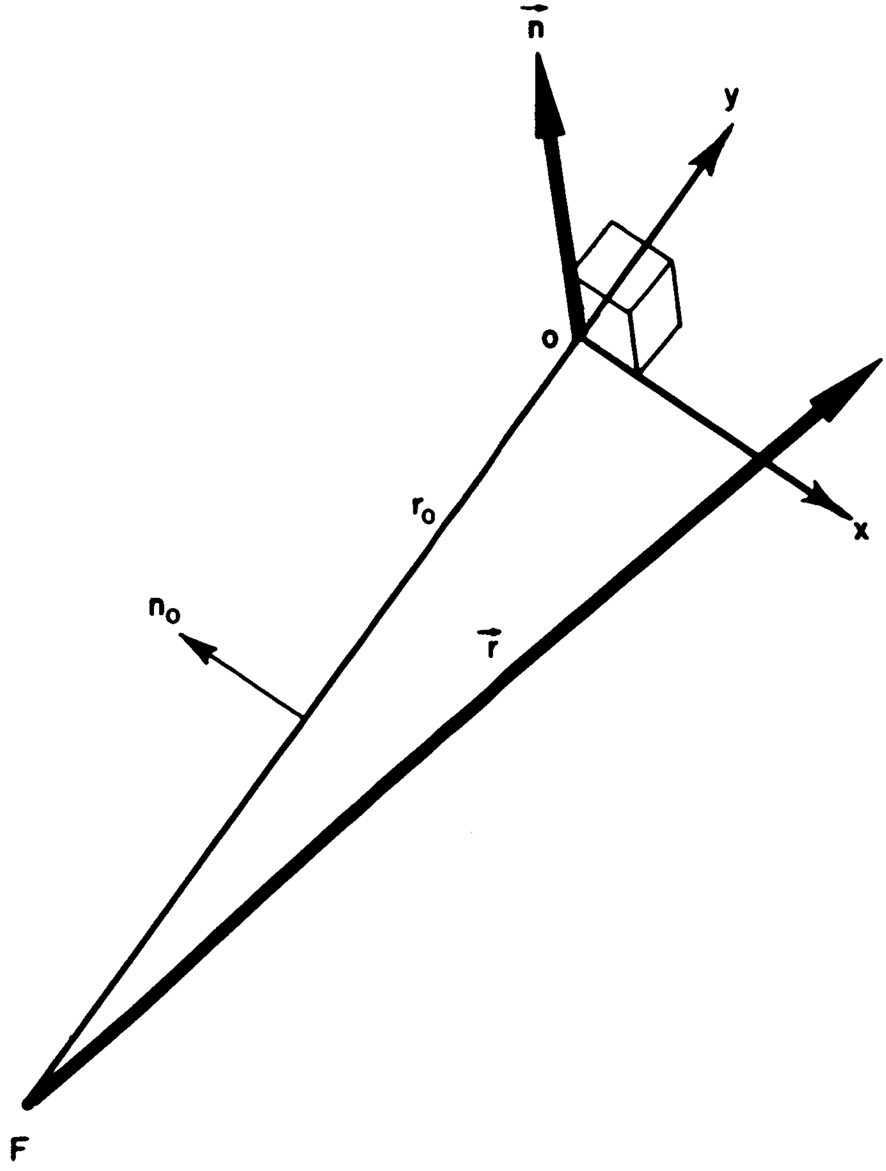


FIG. 25

RELATIONSHIP BETWEEN ROTATING AND NON-ROTATING COORDINATE SYSTEMS



REPUBLIC AVIATION CORPORATION

APPROXIMATE INITIAL VALUES OF LAGRANGE MULTIPLIERS FOR THE TWO POINT BOUNDARY VALUE PROBLEM

By

Jack Richman

Farmingdale, L.I., New York

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TABLE OF CONTENTS

DEFINITION OF SYMBOLS

μ	Gravitational constant
$\underline{R}$	Vehicle position vector
r	$\lvert \underline{R} \rvert$ = magnitude of $\underline{R}$
$\underline{V}$	Velocity vector of vehicle
$\Delta \underline{V}$	Impulse velocity vector
ΔV	$\lvert \Delta \underline{V} \rvert$ = magnitude of $\Delta \underline{V}$
k	Magnitude of thrust
$\underline{T}$	Unit vector in direction of thrust
m	Mass of vehicle
$\dot{m}$	Mass flow
c	Constant, proportional to specific impulse
$\underline{\lambda}$, $\dot{\underline{\lambda}}$, σ	Lagrange multipliers or adjoint variables
λ	$\lvert \underline{\lambda} \rvert$ = magnitude of $\underline{\lambda}$
λ	$\lvert \dot{\underline{\lambda}} \rvert$ = magnitude of $\dot{\underline{\lambda}}$
λ_{ξ}	Component of $\underline{\lambda}$ parallel to $\underline{R}$
λ_{η}	Component of $\underline{\lambda}$ perpendicular to $\underline{R}$
t	Time
t_1	Time at end of first thrust period
t_2	Time at beginning of second thrust period

SUBSCRIPTS

o	Initial value
f	Final value

REPUBLIC AVIATION CORPORATION

Farmingdale, L.I., New York

APPROXIMATE INITIAL VALUES OF LAGRANGE MULTIPLIERS FOR THE TWO POINT BOUNDARY VALUE PROBLEM

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SUMMARY

This report describes a method for obtaining a first estimate of initial values of the Lagrange multipliers for the "Two Point Boundary Value Problem of the Calculus of Variations".

This first estimate is obtained by assuming the "Two Impulse Orbit Transfer Problem" to be a reasonably close approximation to the Calculus of Variations problem.

INTRODUCTION

The method used to solve the two point boundary value problem of the calculus of variations is one where the decision functions are such that all the trajectories being used are extremals [1]. In addition to the state variables, that appear in the equations of motion, there are a number of adjoint variables or Lagrange multipliers that satisfy additional equations for the optimization of the given system. The boundary conditions for the adjoint variables define the natural end-point conditions of the state variables. This natural end point, in general will not be the desired end point. A differential correction scheme provide the means of obtaining another optimum trajectory, the natural end point of which will be closer to the desired end point [2].

The equations of motion of the vehicle in the gravitational field of a single body subject to thrust are as follows:

$$\ddot{\underline{R}} = -\frac{\mu \underline{R}}{r^3} + \frac{k}{m}\underline{T} \tag{1}$$

$$m\,(t_B) = m\,(t_A) + \int_{t_A}^{t_B} \dot{m}\,dt \tag{2}$$

where $\dot{m} = -\frac{k}{c}$ and $\underline{T}$ is a unit vector parallel to the direction of thrust.

The optimum decision functions are determined with the help of the Lagrange multipliers, $\underline{\lambda}$, $\dot{\underline{\lambda}}$, and σ which satisfy the following equations

$$\ddot{\underline{\lambda}} = -\frac{\mu \underline{\lambda}}{r^3} + \frac{3\mu(\underline{\lambda}\ \ \underline{R})\,\underline{R}}{r^5} \tag{3}$$

$$\sigma\,(t_B) = \sigma\,(t_A) + \int_{t_A}^{t_B} \dot{\sigma}\,dt \tag{4}$$

where

$$\dot{\sigma} = \frac{k\lambda}{m^2} \cdot$$

The thrusting program is determined by the sign of the switching function S, which is given by

$$S = \left(\frac{\lambda}{m} - \frac{\sigma}{c}\right) \qquad \begin{matrix} > 0 & k = k_{max} \\ < 0 & k = k_{min} \end{matrix} \tag{5}$$

The direction of the unit thrust vector $\underline{T}$ is given by the direction of the Lagrange multiplier $\underline{\lambda}$

$$\underline{T} = \frac{\underline{\lambda}}{\lambda} \tag{6}$$

The natural end point if reached when

$$\sigma\ (t_F) = 1 \tag{7}$$

The problem is to generate a set of initial values of the Lagrange multipliers such that an optimum orbit can be computed, where the natural end point matches the desired end point. This is accomplished by obtaining a first estimate of the initial values and improving these by using a differential correction scheme.

One of the requirements necessary for a rapid convergence of the differential correction scheme is that the first estimate of the initial values of the Lagrange multipliers be reasonably close. The following is a method for obtaining a first crude estimate of the initial values of the Lagrange multipliers.

INITIAL VALUES OF LAGRANGE MULTIPLIERS

First Method

A first estimate for the initial values of the Lagrange multipliers can be obtained by making the following assumptions about the trajectory.

(a) Two burning periods are required to accomplish the optimum trajectory, one occurring in the time interval t_o to t_1 and the other in the time interval t_2 to t_f. During the time interval t_1 to t_2 the vehicle is in a coasting region.

(b) The time intervals in the thrust regions are so small that $\Delta \underline{V}(t_o)$ and $\Delta \underline{V}(t_F)$ are obtained by solving the "two-impulse orbit transfer" problem, where

$$\begin{aligned} \Delta \underline{V}(t_o) &= \underline{V}(t_1) - \underline{V}(t_o) \\ \Delta \underline{V}(t_f) &= \underline{V}(t_f) - \underline{V}(t_2) \end{aligned} \tag{8}$$

(c) In the regions of thrust the gravitational force may be neglected.

If in addition we assume that the thrust direction is fixed the differential equations for the state variables and the Lagrange multipliers, within the burning region reduce to

$$\dot{\underline{V}} = -\frac{c\dot{m}}{m}\underline{T} \tag{9}$$

$$\ddot{\underline{\lambda}} = 0 \tag{10}$$

$$\sigma(t) = \sigma(t_A) + \int_{t_A}^{t} \dot{\sigma}\,dt \tag{11}$$

where

$$\dot{\sigma} = -\frac{c\dot{m}\lambda}{m^2} \tag{12}$$

and

$$m(t) = m(t_A) + (t - t_A)\,\dot{m} \tag{13}$$

In the burning regions the thrust vector is in the direction of $\Delta \underline{V}$. Therefore from Eq. (6) we have

$$\underline{\lambda} = \lambda \frac{\Delta \underline{V}}{\Delta V} \tag{14}$$

In the coasting region, m and σ are constant. Thus, it follows that

$$\sigma(t_1) = \sigma(t_2) \tag{15}$$

$$m(t_1) = m(t_2) \tag{16}$$

For the computations of the initial values of the Lagrange multipliers, one proceeds as follows:

First Eqs. (9) and (10) are integrated in the two burning regions t_o to t_1 and t_2 to t_f, resulting in

$$m(t_1) = m(t_o)\, e^{-\frac{\Delta V_o}{c}} \tag{17}$$

$$m(t_f) = m(t_o)\, e^{-\frac{(\Delta V_o + \Delta V_f)}{c}} \tag{18}$$

$$\dot{\underline{\lambda}}(t_1) = \dot{\underline{\lambda}}(t_o) = \text{constant} \tag{19}$$

$$\dot{\underline{\lambda}}(t_2) = \dot{\underline{\lambda}}(t_f) = \text{constant} \tag{20}$$

$$\underline{\lambda}(t_1) = \underline{\lambda}(t_o) + (t_1 - t_o)\, \dot{\underline{\lambda}}(t_o) \tag{21}$$

$$\underline{\lambda}(t_f) = \underline{\lambda}(t_2) + (t_f - t_2)\, \dot{\underline{\lambda}}(t_2) \tag{22}$$

where the time spent in the two burning regions is computed by using Eqs. (13), (16), (17), and (18), and is given by

$$(t_1 - t_o) = \frac{m(t_o)\left(e^{-\frac{\Delta V_o}{c}} - 1\right)}{\dot{m}} \tag{23}$$

$$(t_f - t_2) = \frac{m(t_o)\, e^{-\frac{\Delta V_o}{c}} \left(e^{-\frac{\Delta V_f}{c}} - 1\right)}{\dot{m}} \tag{24}$$

From the assumption that the thrust direction is fixed during each burning interval it is evident that $\underline{\lambda}$ and $\underline{\dot{\lambda}}$ are in the same direction. Therefore only the magnitude of $\underline{\lambda}$ and $\underline{\dot{\lambda}}$ need be considered, i.e. λ and $\dot{\lambda}$.

At the transition times t_1 and t_2 the switching function must be zero. Thus,

$$\frac{\lambda(t_1)}{m(t_1)} = \frac{\sigma(t_1)}{c} \tag{25}$$

and

$$\frac{\lambda(t_2)}{m(t_2)} = \frac{\sigma(t_2)}{c} \tag{26}$$

It can be shown that by integrating Eq. (11) in the two burning regions and making use of Eqs. (12) through (26) one forms the following three independent equations with five unknowns, i.e., $\sigma(t_o)$, $\lambda(t_o)$, $\dot{\lambda}(t_o)$, $\lambda(t_f)$ and $\dot{\lambda}(t_f)$

$$\frac{c}{m(t_o)} - \sigma(t_o) - \frac{\Delta V_o}{\dot{m}}\, \dot{\lambda}(t_o) = 0 \tag{27}$$

$$\frac{c\, e^{\frac{\Delta V_o}{c}}}{m(t_o)}\, \lambda(t_f) + \frac{c}{\dot{m}}\left(1 - e^{-\frac{\Delta V_f}{c}}\right) \dot{\lambda}(t_f) + \frac{c}{m(t_o)}\, e^{\frac{\Delta V_o}{c}} \left(e^{\frac{\Delta V_f}{c}} - 1\right) \lambda(t_o)$$

$$+ \frac{\dot{\lambda}(t_o)}{\dot{m}} \left[c\left(1 - e^{\frac{\Delta V_f}{c}}\right) + \Delta V_f\right] = 1 \tag{28}$$

$$\frac{e^{\frac{\Delta V_o}{c}}}{m(t_o)}\left[\lambda(t_o)-\lambda(t_f)\right]+\frac{1}{\dot{m}}\left(1-e^{\frac{\Delta V_o}{c}}\right)-\frac{1}{\dot{m}}\left(1-e^{-\frac{\Delta V_f}{c}}\right)\dot{\lambda}(t_f)=0 \qquad (29)$$

By making use of the transversality condition $\underline{\lambda}\cdot\dot{\underline{R}}-\dot{\underline{\lambda}}\cdot\dot{\underline{V}}+\sigma\dot{m}=0$ at times t_o and t_f one can obtain two more equations.

$$-\dot{\lambda}(t_o)\,\frac{\underline{V}(t_o)\cdot\Delta\underline{V}_o}{\Delta V_o}-\frac{c\dot{m}}{m(t_o)}\,\lambda(t_o)+\sigma(t_o)\,\dot{m}=0 \qquad (30)$$

$$-\dot{\lambda}(t_f)\,\frac{\underline{V}(t_f)\cdot\Delta\underline{V}_f}{\Delta V_f}-\frac{c\dot{m}}{m(t_o)}\,e^{\frac{\Delta V_o+\Delta V_f}{c}}\,\lambda(t_f)+\dot{m}=0 \qquad (31)$$

Eqs. (27) through (31) constitute five equations with five unknowns. The solution of this system of equations is given by

$$\underline{\lambda}(t_o)=\frac{m(t_o)}{c}\,e^{-\frac{(\Delta V_o+\Delta V_f}{c}}\,\frac{\Delta\underline{V}_o}{\Delta V_o} \qquad (32)$$

$$\dot{\underline{\lambda}}(t_o)=0 \qquad (33)$$

$$\sigma(t_o)=e^{-\frac{(\Delta V_o+\Delta V_f)}{c}} \qquad (34)$$

$$\underline{\lambda}(t_f)=\frac{m(t_o)}{c}\,e^{-\frac{(\Delta V_o+\Delta V_f)}{c}}\,\frac{\Delta\underline{V}_f}{\Delta V_f} \qquad (35)$$

$$\dot{\underline{\lambda}}(t_f)=0 \qquad (36)$$

It is of interest to note that the magnitudes of $\underline{\lambda}$ at the initial and final times are equal and directly proportional to the mass at the final time. In addition, the value of σ is also proportional to the final mass and may be expressed as

$$\sigma(t)=\frac{m(t_f)}{m(t)} \qquad (37)$$

Second Method

An approach for obtaining a better first approximation is to remove or at least "relax" some of the assumptions made in the first method. More specifically, instead of completely neglecting the gravitational force in the regions of thrust it can be assumed that the gravitational force has a constant value of $\frac{-\mu \underline{R}_o}{r_o^3}$ in the first region and $\frac{-\mu \underline{R}_f}{r_f^3}$ in the second region.

In addition, we assume that the direction of the total acceleration in the two regions of thrust is parallel to the vector $\Delta \underline{V}_o$ and $\Delta \underline{V}_f$, respectively. This implies that the direction of the thrust is not fixed.

It is clear that in the region of thrust the vector $\underline{\lambda}$ lies in the plane formed by the vectors $\underline{R}$ and $\Delta \underline{V}$. It is most convenient to resolve $\underline{\lambda}$ into components along the vector $\underline{R}$ and normal to it. These two components are designated as λ_ξ and λ_η, respectively.

The differential equation for $\underline{\lambda}$ can now be written as

$$\ddot{\lambda}_\xi = \frac{2\mu}{\gamma^3} \lambda_\xi \tag{38}$$

$$\ddot{\lambda}_\eta = \frac{\mu}{r^3} \lambda_\eta \tag{39}$$

The solution to Eqs. (38) and (39) is given by

$$\lambda_\xi = \lambda_\xi(t_o) \cosh \sqrt{\frac{2\mu}{r^3}}\, t + \sqrt{\frac{r^3}{2\mu}}\, \dot{\lambda}_\xi(t_o) \sinh \sqrt{\frac{2\mu}{r^3}}\, t \tag{40}$$

$$\lambda_\eta = \lambda_\eta(t_o) \cos \sqrt{\frac{\mu}{r^3}}\, t + \sqrt{\frac{r^3}{\mu}}\, \dot{\lambda}_\eta(t_o) \sin \sqrt{\frac{\mu}{r^3}}\, t \tag{41}$$

Since the intervals of thrust are assumed to be of short duration it is permissible to approximate Eqs. (40) and (41) in the regions of thrust by neglecting the second order terms of a Taylor series expansion, i.e.,

$$\underline{\lambda}(t) \approx \underline{\lambda}(t_o) + (t - t_o)\, \dot{\lambda}(t_o) \qquad t_o \le t \le t_1 \tag{42}$$

$$\underline{\lambda}(t) \approx \underline{\lambda}(t_2) + (t - t_2)\, \dot{\lambda}(t_2) \qquad t_2 \le t \le t_f \tag{43}$$

Similarly, one can approximate $\dot{\lambda}$ in the regions of thrust to the same order of accuracy.

$$\dot{\lambda}_\xi(t) \approx \frac{2\mu}{r^3}(t - t_o)\lambda_\xi(t_o) + \dot{\lambda}_\xi(t_o) \tag{44}$$

$$t_o \le t \le t_1$$

$$\dot{\lambda}_\eta(t) \approx -\frac{\mu}{r^3}(t - t_o)\lambda_\eta(t_o) + \dot{\lambda}_\eta(t_o) \tag{45}$$

$$\dot{\lambda}_\xi(t) \approx \frac{2\mu}{r^3}(t - t_2)\lambda_\xi(t_2) + \dot{\lambda}_\xi(t_2) \tag{46}$$

$$t_2 \le t \le t_f$$

$$\dot{\lambda}_\eta(t) \approx -\frac{\mu}{r^3}(t - t_2)\lambda_\eta(t_2) + \dot{\lambda}_\eta(t_2) \tag{47}$$

The procedure for obtaining the initial values of the Lagrange multipliers is now the same as in the first method except that Eqs. (19) through (22) are now replaced by Eqs. (42) through (47).

CONCLUSION

A set of approximate initial values of the Lagrange multipliers have been derived. In addition, a method for obtaining a better first approximation has been outlined. It should be pointed out, however, that as one attempts to obtain these improved first approximations in the manner outlined, the algebraic manipulation of the expressions involved become more cumbersome and additional approximations may be needed.

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SPACE SCIENCES LABORATORY

MECHANICS SECTION

THE TWO-VARIABLE EXPANSION METHOD
FOR LUNAR TRAJECTORIES

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SUMMARY

This report is submitted in partial fulfillment of the contract in "Space Flight and Guidance Theory," No. NAS8-11040. It presents a discussion of Lagerstrom and Kevorkian's two-variable expansion method for the computation of lunar trajectories. Section 2 discusses the general background of the method in terms of singular perturbation theory. Section 3 discusses the major steps in the development of a uniformly valid solution for earth-moon trajectories and Section 4 presents a slightly different approach to the same problem.

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1. INTRODUCTION

In refs. (3) and (4) a new method was suggested by Lagerstrom and Kevorkian for the computation of lunar trajectories. The method was similar to one which had been used successfully in a number of singular perturbation problems of boundary layer theory (refs. 1, 2). The result of approaching the lunar trajectory problem as a singular perturbation problem was a uniformly valid solution (i. e. valid everywhere in the earth-moon space) to first order in the parameter for a certain class of trajectories. The class of trajectories is that which starts in a neighborhood of order μ near the earth and arrives near the moon to within a neighborhood of order μ. Similarly to other singular perturbation problems, this uniformly valid solution was obtained by formulating two solutions, one valid near the earth (the "outer solution") and the other valid near the moon (the "inner solution"). The inner solution is expressed in terms of "blown up" variables. The outer and inner solutions are left undetermined by introducing a number of constants; these constants are determined such that the singularities in the outer and inner solutions cancel when they are combined to form the "composite solution."

The basic idea of the method was worked out in its application to the two-fixed center problem with special initial conditions (ref. 3); then the same technique was used in the restricted three body problem with more general initial conditions (ref. 4). One of the most interesting results was the finding that the outer solution must contain a part which is proportional to the small parameter μ , or else it cannot be matched to the inner solution; the outer solution can thus be interpreted as an earth centered Kepler ellipse with a first order correction to take care of the moon's perturbation. In comparing this method with the usual way of "patching conics", it was thus stated that a patched conic method could not be accurate, unless the geocentric ellipse were corrected for the moon's perturbation. The two-variable expansion method was thus offered as an improvement over patched conic methods and it appeared to be (at least initially) equally practical.

This report presents an explanation of the method (in Section 3), based mostly on ref. 4, and the beginning of a somewhat different approach (in Section 4). The claim that this report is an "explanation" is made with all modesty; it is an explanation in the sense that it presents and discusses the major steps of the developments in ref. 4, leaving out many of the laborious details. In this way it is hoped that the reader may gain a full appreciation and understanding of this very interesting method; this report may thus serve as an introduction to the reading of refs. 3 and 4. This explanation is preceded (Section 2) by a general discussion of singular perturbation theory, based mostly on ref. 1 and 2. In particular with respect to this section, and the conjecture and theorem on which the discussion is based, the authors gratefully acknowledge personal communication with Dr. Kevorkian.

In Section 4 the beginning of a slightly different approach to the same problem is presented. Whereas the work by Lagerstrom and Kevorkian is formulated in inertial coordinates, this new approach makes use of rotating coordinates, and the Jacobi Integral in order to solve the problem as a third order system of differential equations.

2. DISCUSSION OF THE TWO-VARIABLE EXPANSION METHOD

The method used by Lagerstrom and Kevorkian to formulate a uniformly valid representation of earth-moon trajectories is that which is used in the singular perturbation problems of boundary layer theory. A singular perturbation problem may be characterized as follows: a differential equation $L(x, u, \varepsilon) = 0$ and boundary conditions $B(u, \varepsilon) = 0$ depend on a small positive parameter ε in such a way that the order or type of L change when $\varepsilon = 0$, while the number of boundary conditions remains unchanged. Thus, if u^0 represents the solution of $L(x, u, 0) = 0$, one may not expect that u approaches u^0 uniformly as $\varepsilon \longrightarrow 0$.

Fundamental to the solution of singular perturbation problems is the introduction of certain limits. Consider functions f of ε, positive and continuous in $0 < \varepsilon < A$ and tending to a definite limit as $\varepsilon \longrightarrow 0$; introduce a new variable $x_f = \dfrac{x}{f}$, then a limit on $F(x, \varepsilon)$ is defined as

$$\lim_f F(x, \varepsilon) = \lim_{\varepsilon \to 0} F\{(f\, x_f), \varepsilon\}, \quad x_f \text{ fixed and } \neq 0.$$

If $f = 1$, the limit is usually called "outer limit," and x the "outer variable", since in the boundary layer problem which motivated this formulation this limit presents a satisfactory approximation in the physical space away from the boundary. An "inner variable" and "inner limit" are obtained in many problems by putting $f = \varepsilon$; the inner limit is an approximation in that region of the physical space where the differential equation changes order (or type) as $\varepsilon \longrightarrow 0$. As the inner variable is kept constant, the physical variable x tends to $\emptyset$ as $\varepsilon \longrightarrow 0$; it is as if the problem is discussed in terms of "stretched" or "blown-up" variables. Theoretically of great importance are also the concepts of "intermediate variable" and "intermediate limit," which are intuitively understood as obtained by a function $f(\varepsilon)$, where the order of magnitude $O(f(\varepsilon))$ is in between $O(1)$ and $O(\varepsilon)$. A more rigorous discussion is given by Kaplun in ref. 1.

The formulation of a solution based on inner and outer limit is based on a "matching" of the two limits. But since there is no a-priori reason why the regions of validity of inner and outer limits should overlap, it may seem to be surprising that this has been so successful in many problems. It is here that Lagerstrom and Kaplun have contributed greatly to the understanding of the problem by using the intermediate expansion to bridge the gap. In ref. 5 Erdelyi discusses this in some more detail, but (as here) also in an intuitive manner.

The method by which a uniformly valid solution of singular perturbations is obtained is based on a conjecture and a theorem. The conjecture is: the solution of the limiting differential equation (obtained by subjecting the differential equation to the above defined limiting process) is identical with the limiting approximation of the exact solution. Thus, if an exact solution cannot be obtained directly, one can get an approximation (actually an asymptotic expansion) by solving the limiting differential equation. The validity of this conjecture is supported by a number of problems to which exact solutions are available.

In a singular perturbation problem it will be necessary to combine at least two limiting solutions (i.e. inner and outer) to obtain a uniformly valid solution, that is a solution valid in the entire physical space of the variables. Kaplun's extension theorem bridges the gap which may exist between the regions of validity of the limiting solutions. The formulation of the extension theorem requires the definition of "equivalence classes". Let f and g be functions of $\mathcal{E}$, positive and continuous and tending to a definite limit as $\mathcal{E} \longrightarrow 0$, then $f(\mathcal{E})$ and $g(\mathcal{E})$ belong to the same equivalence class if

$$0 < \lim_{\mathcal{E} \to 0} \frac{f}{g} < \infty$$

A partial ordering of equivalence classes is defined by

$$\operatorname{ord} f < \operatorname{ord} g \text{ if } \lim_{\mathcal{E} \to 0} \frac{f}{g} = 0.$$

A set S of equivalence classes is convex if, for every ord f and ord g in S, ord $f <$ ord $h <$ ord g implies ord h is in S. Open and closed convex sets of equivalence classes are defined according to the usual definitions of set theory. The extension theorem may now be formulated as:

If an approximation is valid to order $\mathcal{E}$ in a closed set S its domain of validity may be extended to an open convex set S, containing S.

Thus, the inner and outer expansions are valid in larger regions than those for which they were derived. The regions of validity of inner and outer expansions may now overlap or else they may be joined by an intermediate expansion. Whether the inner and outer expansions are matched directly or by the use of an intermediate expansion, the matching is performed by using overlapping regions of validity provided by the extension theorem. It will be seen that in the earth-moon trajectory problem the matching can be performed directly without the use of an intermediate expansion.

The following illustration may be of some help in understanding the meaning of the expansion theorem. In figure 1 the shaded areas in the x, ε space indicate the regions of validity of inner and outer expansions in a problem with singularity at $x = 0$.

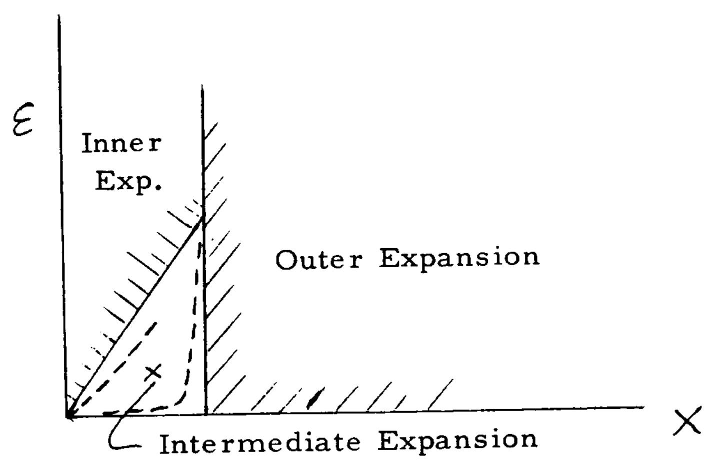


Fig. 1 EXTENSION THEOREM

The outer expansion is valid for a range of x bounded away from zero. The region for the inner expansion shows the typical behavior near the singularity: As ε tends to zero the physical variable x tends to zero also; the inner variable $x_\varepsilon = \frac{x}{\varepsilon}$ remains finite. It is clear that for small ε the regions of validity of inner and outer expansions do not overlap. But the expansion theorem provides for small additional regions of validity, indicated by the dashed lines in fig. 1. These regions can now be used to provide overlap with an intermediate expansion (obtained by introducing the intermediate variable $x_f = \frac{x}{f(\varepsilon)}$, ord $\varepsilon <$ $f(\varepsilon) <$ ord 1) and matching can be performed.

The plan for formulating a uniformly valid solution of a singular perturbation problem is now clear. An outer solution of the differential equation is obtained, satisfying some of the boundary conditions. Typically, the boundary conditions near the singularity are neglected, but the outer solution must have as many arbitrary constants as there are neglected boundary conditions. Next, the problem is "blown up" in the region near the singularity by the transformation to inner variables. The boundary conditions which were neglected in the outer solution can now be satisfied by the inner solution, but the other boundary conditions will in general not make sense. Therefore the inner solution is partly indeterminate. To remove this indeterminacy the inner and outer solutions are "matched" as follows. The outer solution is evaluated at the inner region, the inner solution is evaluated at the outer region and these two functions are equated after the introduction of the transformation $x_f = \frac{x}{f(\varepsilon)}$. Finally, a "composite solution" is obtained by adding the inner and outer solutions and subtracting <u>either</u> the inner solution evaluated at the outer region <u>or</u> the outer solution evaluated at the inner region. This either/or condition reflects of course just the matching condition. The matching and the formulation of a composite solution described here is possible when the regions of validity of inner and outer expansions overlap, if this is not the case the same procedures have to be followed on either side of an intermediate expansion.

The extension theorem is the basis for success in matching; the conjecture makes it plausible that the composite solution is uniformly valid, even though the inner and outer solutions themselves are only valid in their respective regions.

The application of these principles to the earth-moon trajectory problem takes the following form. The equations of motion of the planar restricted three body problem (in non-rotating coordinates) are formulated with one of the coordinates, x , as the independent variable. Uniformly valid expressions are sought for the time and the other coordinate as functions of x and the small parameter μ , the earth-moon mass ratio. Near the earth the influence of the moon is seen in the equations of motion as a perturbation (proportional to μ) of the Kepler equations. Clearly, in this problem the singularity is located at $x \to 1$, since near the moon the attraction of the moon itself is the major force. An outer solution is formulated in the physical variables x , t and y ; it describes the earth-centered part of the trajectory. An inner solution is formulated in the "blown-up" variables $\bar{x}$, $\bar{t}$ and $\bar{y}$, the differential equations for which show the moon's attraction as the major force. In principle the outer and inner solutions are asymptotic expansions of which the separate terms can be obtained by substituting $t = t_0 + \mu t_1 + \mu^2 t_2 + \ldots,$ $y = y_0 + \mu y_1 + \mu^2 y_2 + \ldots,$

in the equations of motion, ordering the results according to powers of μ and solving the equations for $t_0, y_0, t_1, y_1, \ldots$ in succession. A major result of Lagerstrom's and Kevorkian's investigation was the finding that, in order to formulate a first order solution, the outer solution must contain the correction of order μ to the earth-centered Kepler trajectory. The reason is that the angular momentum near the moon (for a passage at distance of order μ) is of order μ , and can thus only be defined when terms of order μ are included in the approach trajectory. The matching of inner and outer solutions is performed by equating term by term the results of evaluating the outer solution at $X = 1$ and the inner at $\bar{X} = -\infty$; for this purpose the inner as well as the outer solution are expressed in the inner variable. The results of the matching are the elements of the moon-centered hyperbola and the phase constant of the moon. The composite solution is obtained by adding the inner and outer solutions and subtracting the outer expansion of the inner solution. From the form of inner and outer solutions it is clear that no intermediate solution is required.

In their first paper on the three-body problem (ref. 8) Lagerstrom and Kevorkian treated the problem of two fixed force centers (the Euler problem). They discussed trajectories which leave from the center of the larger mass, the Kepler part of the outbound trajectory being a straight line. The major result was that 1) a uniformly valid solution to order μ could indeed be obtained and 2) the outer solution must contain a correction of order μ in order to be able to determine the constants of the inner solution. Because of the very special initial conditions the outer and inner expansions are of simple form and therefore the principles of the method are clearly demonstrated. In their second paper (ref. 9) they treated the more practical restricted three body problem with arbitrary initial conditions (although restricted to a neighborhood of order μ near the earth). While following the same method in principle, the details of the analysis are somewhat obscured by the added difficulties from the more general initial conditions and the motion of the moon. The following section refers in particular to this paper; it interprets and explains the method by lifting out the essential difficulties and omitting all easily understood details. References 10 and 11 discuss some numerical aspects.

The following section contains an outline and discussion of ref. 9. It is hoped that, by concentrating on the major difficulties, that section, together with the general discussion in this section, will be useful for the better understanding and appreciation of the very interesting method of Lagerstrom and Kevorkian.

3. TWO-VARIABLE EXPANSION METHOD FOR EARTH-MOON TRAJECTORIES

3.1 Equations of Motion; Outer and Inner Variables

In geocentric, non-dimensional, inertial coordinates x, y the equations of motion for the planar restricted three body problem are:

$$\ddot{x} + (1-\mu)\frac{x}{r^3} = \mu f \qquad f = \frac{\xi_m - x}{r^3} - \xi_m \tag{1}$$

$$\ddot{y} + (1-\mu)\frac{y}{r^3} = \mu g \qquad g = \frac{\eta_m - y}{r^3} - \eta_m$$

where

$$r^2 = x^2 + y^2$$

μ is the earth-moon mass ratio, $\mu = \dfrac{M_m}{M_E + M_m}$, and the coordinates of the moon are

$$\xi_m = \cos(t-T), \quad \eta_m = \sin(t-T) \tag{2}$$

T is a phase angle which is to be determined later.

The goal is to formulate uniformly valid expressions (i.e. valid near the earth as well as near the moon) to order μ for trajectories which leave from a neighborhood of order μ near the earth and reach a neighborhood of order μ near the moon.

The outer variables, to be used for the outbound trajectory near the earth, are the physical variables x, y, t. The inner variables will be chosen as

$$\bar{x} = \frac{x - \cos(t-T)}{\mu}, \quad \bar{y} = \frac{y - \sin(t-T)}{\mu} \tag{3}$$

$$\bar{t} = \frac{(t-T) - \tau}{\mu}$$

This choice assures that the motion near the moon is Keplerian up to and including the first order of μ and that the velocity far from the moon is of the same order (i.e. of order 1) as the velocity far from the earth. The additional phase angle τ is introduced so that $\bar{t}$ can be made to vanish at perilune.

It is interesting to note that if a scale factor of $\mu^{-1/3}$ is used in the definition of $\bar{X}$ and $\bar{Y}$ and the time is left unscaled, the equations of motion in terms of $\bar{X}$, $\bar{Y}$ and $\bar{t}$ after letting $\mu \rightarrow 0$ are the Hill equations; these equations are valid in Hill's region, i.e. a region of order $\mu^{1/3}$ near the moon. If an intermediate solution were required, these equations could provide it. It will be seen that the inner and outer solutions can be matched without using an intermediate solution, although this cannot be expected a priori. Apparently, for the class of trajectories considered here (i.e. coming from a neighborhood of order μ near the moon), the passage through Hill's region is so fast that Hill's equations do not need to be considered.

It will be convenient to introduce the coordinate X as the independent variable; the matching of inner and outer solutions is then done on the basis of distance instead of time. The equations of motion in the outer variables are then

$$
\begin{aligned}
-\frac{t''}{t'^3} + (1-\mu)\frac{X}{r^3} &= \mu f \\
\frac{Y''}{t'^2} - \frac{t''Y'}{t'^3} + (1-\mu)\frac{Y}{r^3} &= \mu g
\end{aligned}
\tag{4}
$$

The equations of motion in inner variables, valid near the moon, are Keplerian up to and including the first order of μ and do not have to be written here. Terms proportional to the first power of μ are not present because of the scaling of the variables and because the moon centered $\bar{x}$, $\bar{y}$ axes are taken parallel to the earth centered x, y axes.

3.2 Outer Expansion

The right hand sides of equs. (4) represent small perturbations due to the moon; near the earth the solution of equ. (4) is thus nearly Keplerian and it will be convenient to specify the initial conditions of the trajectory by giving the values of the Kepler integrals. The integrals to be chosen are the total energy h_e, the angular momentum ℓ_e the location of perigee and the time of perigee passage. In order to reach the neighborhood of the moon, the total energy must be $O(1)$; the initial velocity is thus $O(\mu^{-\frac{1}{2}})$ and, since the trajectory leaves from a neighborhood of order μ near the earth, the angular momentum is $O(\mu^{\frac{1}{2}})$. Without loss of generality the perigee may be taken to be on the x-axis (on the side of the earth opposite to that of the moon). The initial conditions are thus

$$\text{at} \quad x=0 : h_e = -\rho^2 \tag{5}$$

$$\ell_e = \mu^{1/2}\lambda \tag{6}$$

perigee on x-axis

and the time is specified by requiring that the Keplerian approximation is exact at $x=0$ to all orders of μ.

Since the angular momentum is of order $\mu^{1/2}$ it is clear that, for the class of trajectories discussed here, y is also of order $\mu^{1/2}$. The asymptotic expansions for t and y may thus be taken to be

$$t(x,\mu) = t_0(x,\mu) + \mu t_1(x) + \dots \tag{7}$$

and

$$y(x,\mu) = \mu^{\frac{1}{2}} y_{\frac{1}{2}}(x,\mu) + \mu y_1(x) + \dots \tag{8}$$

The differential equations for t_0, t_1, $y_{\frac{1}{2}}$ and y_1 are found by substituting (7) and (8) into the equations of motion (4) and by ordering the results according to powers of μ. The equations for t_0 and $y_{\frac{1}{2}}$ are of course just the Keplerian equations (equ. 4 with zero in the right handsides) and their solutions do not have to be repeated here. However, one detail must be pointed out. Whenever the parameter μ appears as $(1-\mu)$, the nondimensional gravitational constant, it is not subjected to the limit process. Furthermore, the angular momentum constant has been written as $\mu^{\frac{1}{2}}\lambda$ and for these two reasons the parameter μ appears thus in the expressions for the Keplerian part of the trajectory. This seems at first to be in contradiction with the principle of the singular perturbation method according to which the zero-order solution would be independent of the small parameter. Allowing the small parameter to appear in the Keplerian part results in somewhat more convenient expressions. The first part of $t(x,\mu)$ is now written as

$$t_c(x,\mu) = t_{00}(x) + \mu t_{01}(x) \tag{9}$$

If the solution had been started with $t_{00}(X)$, according to a strict application of the limit process it would be necessary to consider a separate "boundary layer" near the earth, because the relative orders of magnitude of the terms in $t_0(X, \mu)$ are different for $X = O(1)$ and $X = O(\mu)$. This nonuniformity has nothing to do with the moon's perturbation and is taken care of by letting μ appear in the Keplerian solution.

The equations for the first order corrections t_1 and y_1 are:

$$- \frac{t_1''}{t_{00}'^3} + \frac{3 t_{00}'' t_1'}{t_{00}'^4} = f_0 \tag{10}$$

$$\frac{y_1''}{t_{00}'^2} - \frac{t_{00}'' y_1'}{t_{00}'^3} + \frac{y_1}{X^3} = g_0 \tag{11}$$

with $f_0 = \{f(x)\}_{\mu=0}$ and $g_0 = \{g(x)\}_{\mu=0}$.

Because the initial conditions have been chosen such that the Kepler solution is exactly valid at $X = 0$, the initial conditions for t_1 and y_1 are simply

$$t_1(0) = y_1(0) = 0 \quad \text{and} \quad t_1'(0) \quad \text{and} \quad y_1'(0).$$

3.3 First Order Corrections in Outer Expansion

The first order integrals of (10) and (11) are easily obtained as

$$-t_1' = t_{00}'^3 \int_0^X f_0(\xi)\, d\xi \tag{12}$$

and

$$x y_1' - y_1 = t_{00}' \int_0^X \xi\, t_{00}'(\xi)\, g_0(\xi)\, d\xi \tag{13}$$

In principle t_1 and y_1 are thus obtained by quadratures but so far no analytic expressions for t_1 and y_1 have been found. The functions f_c and g_c are unbounded for $x \longrightarrow 1$ and their behavior near $x = 1$ can be studied by expressing the several parts of f_c and g_c in Taylor series near $x = 1$. The results are

$$f_c(x) = \frac{U}{(1+U^2)^{3/2}}\left[\frac{U^2}{(1-x)^2} + \frac{1}{(1-x)}\right] + \Phi(x) \tag{14}$$

$$g_c(x) = \frac{-1}{(1+U^2)^{3/2}}\left[\frac{U^2}{(1-x)^2} + \frac{1}{(1-x)}\right] + \Gamma(x) \tag{15}$$

where $\Phi(x)$ and $\Gamma(x)$ are the regular parts of f_c and g_c, and

$$U = \sqrt{2(1-\rho^2)} = \frac{1}{t_{co}'(1)} \tag{16}$$

which is the x velocity of the Keplerian trajectory at $x = 1$.

Using (14) and (15) the first order corrections to the Keplerian part of the outer expansion may be written as

$$t_1(x) = -\frac{U}{(1+U^2)^{3/2}}\int_0^x t_{00}'^{3}(\xi)\left[\frac{U^2}{1-\xi} - \log(1-\xi) + \frac{(1+U^2)^{3/2}}{U}\int_0^\xi \Phi(\xi)\,d\xi\right]d\xi \tag{17}$$

$$y_1(x) = -\frac{x}{(1+U^2)^{3/2}}\int_0^x \frac{t_{co}'(\xi)}{\xi^2}\left\{\int_0^\xi \xi\, t_{co}'(\xi)\left[\frac{U^2}{(1-\xi)^2} + \frac{1}{1-\xi} - (1+U^2)^{3/2}\Gamma(\xi)\right]d\xi\right\}d\xi \tag{18}$$

Since (at least to this time) no analytic solutions for t_1 and y_1 have been found, the complete trajectory can only be computed by evaluating the quadratures numerically. Clearly, this causes numerical difficulties because of the singular behavior near $x = 1$. It is of some help in establishing a computer program based on this method that $t_1(x)$ and $y_1(x)$ depend on only one parameter, namely the total energy $-\rho^2$. The corrections could thus be computed and tabularized once and for all. Also, near $x = 1$ t_1 and y_1 may be expressed much more simply as

$$t_1 = (1+u^2)^{-3/2} \log(1-x) + \gamma(\rho) + O(1) \tag{19}$$

$$y_1 = (1+u^2)^{-3/2} \log(1-x) + \delta(\rho) + O(1) \tag{20}$$

where γ and δ are functions of the total energy alone. Unfortunately, γ and δ become unbounded as $\rho \longrightarrow 1$, that is for the minimum energy trajectories. This difficulty has been treated in detail in ref. 6.

Equ. (19) and (20), and particularly the functions γ and δ, play an important role in the matching of outer and inner expansions.

3.4 The Inner Expansion

The equations in the inner variables $\bar{x}$, $\bar{y}$ and $\bar{t}$ have μ only to the second and higher powers. Since the present purpose is to develop a solution to first order in μ the moon centered part of the trajectory is thus Keplerian and, in particular, hyperbolic. It will be convenient to characterize this hyperbola by the four constants

u_1, the $\bar{x}$ component of velocity at $\bar{x} = -\infty$

V_1, the $\bar{y}$ component of velocity at $\bar{x} = -\infty$

$K_1 = \dfrac{AV_1 - Bu_1}{u_1}$ related the direction of the asymptote

and $\bar{t} = 0$ at perilune.

In the definition of K_1 ,

$$A = \bar{a}\,\bar{e}\cos\bar{\epsilon} \qquad B = \bar{a}\,\bar{e}\sin\bar{\epsilon}$$

$\bar{a}$ is the semimajor axis, $\bar{e}$ the eccentricity and $\bar{\epsilon}$ is the counter-clockwise angle between the X axis and the apse line of the hyperbola. The expressions $\bar{y}(\bar{x})$ and $\bar{t}(\bar{x})$ do not have to be given here (since they are Keplerian) except as they are needed for the matching of inner and outer solutions. For this purpose their values as $X \longrightarrow -\infty$ are needed. These are

$$\bar{y} = \frac{V_1}{U_1}\,\bar{x} - \frac{AV_1 - BU_1}{U_1} \tag{21}$$

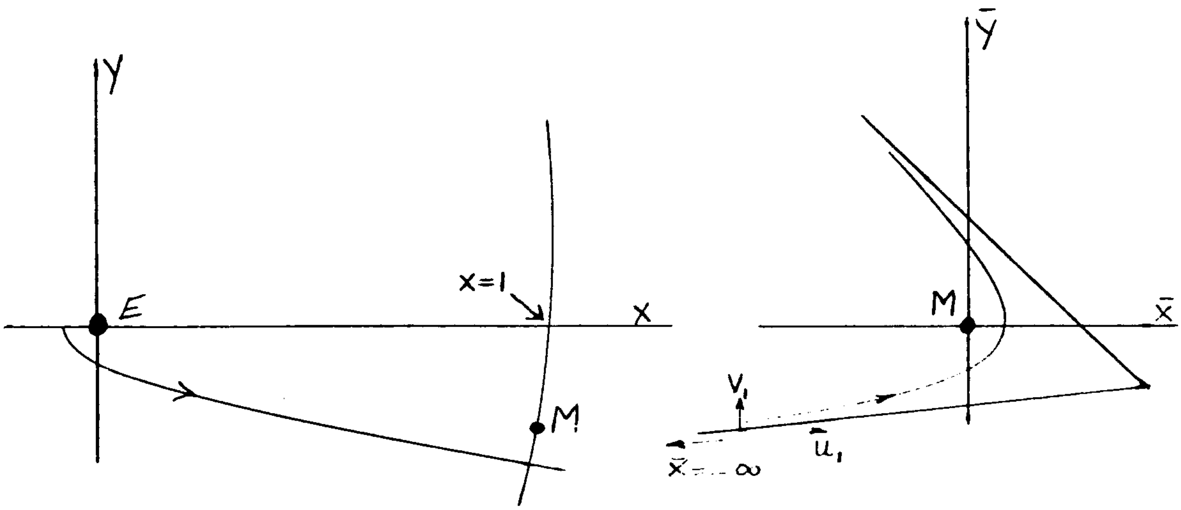
$$\bar{t} = \frac{\bar{x} - A}{U_1} + \bar{a}^{3/2}\log\frac{-2\bar{x}}{U_1\,\bar{a}^{3/2}\,\bar{e}} \tag{22}$$

as follows readily from the equations of hyperbolic motion (most conveniently by letting the eccentric anomaly approach $-\infty$).

3.5 Matching of Inner and Outer Solutions

The purpose of matching the inner and outer expansions is to determine the constants of the moon,centered hyperbola, thereby also relating the singularities in the two expansions in such a way that they cancel each other in the composite solution. Because the singularities are logarithmic in nature in the inner as well as the outer solutions, such matching can apparently be achieved without the use of an intermediate expansion.

The geometry of the matching is illustrated in Fig. 2, as much as it can be illustrated. The part of the figure related to the inner expansion is drawn in the scaled coordinates $(\bar{x}, \bar{y})$ and must be thought as infinitely small in comparison with the figure for the outer expansion. It may be remarked that this matching is strictly analytical, whereas the "patching" of conics is strictly geometrical. A direct comparison of the two methods is therefore difficult; such comparison should be based on the final numerical results.



OUTER EXPANSION

$$t = t_{00}(x) + \mu t_{01}(x) + \mu t_1(x)$$
$$y = \mu^{1/2} y_{\frac{1}{2}}(x) + y_1(x)$$

INNER EXPANSION

$$t = T + \mu \bar{t} + \tau$$
$$y = \mu \bar{y} + \sin(t - T)$$

Figure 2 GEOMETRY OF MATCHING

The matching is performed by evaluating the outer expansion at $x = 1$ and equating the result term by term to the inner expansion evaluated at $\bar{x} = -\infty$, both expansions being expressed in the inner variable. (The important thing is that both expansions are expressed in the same variable;

the present choice of inner variable is simply for the sake of convenience.)

The part of the outer expansion identified with t_{cc} is evaluated near $X = 1$ by writing two terms of its Taylor expansion at $X = 1$:

$$t_{cc}(x) = t_{cc}(1) + (x-1)t'_{co}(1) = t_{cc}(1) + (x-1)\mathcal{U}^{-1}$$

using equ. (16). The inner variable is introduced by $X = \mu\bar{X} + \cos(t-T)$ from equ. (3), and if it is assumed that $(t-T)$ is small (as it will be shown to be), there results

$$t_{cc}(x) = t_{cc}(1) + \mu\left\{\bar{x}\,\mathcal{U}^{-1} - \tfrac{1}{2}(t-T)^2\mathcal{U}^{-1}\right\} \tag{23}$$

The introduction of the inner variable into the expression for t_1 (equ. 19) is taken care of by putting $(1-X) = \mu\bar{X}$, the term $\cos(t-T)$ in equ. (3) being put equal to unity with enough accuracy since t_1 is multiplied by μ. By combining equs. (7), (9), (19) and (23), the outer expansion evaluated near $X = 1$ and expressed in the inner variable is thus

$$\begin{aligned} t(\bar{x}) = t_{cc}(1) &+ \mu\bar{x}\,\mathcal{U}^{-1} - \tfrac{1}{2}(t-T)^2\mathcal{U}^{-1} \\ &+ \mu\left\{t_{01}(1) + (1+\mathcal{U}^2)^{-3/2}\left[\log(-\bar{x}) + \log\mu\right] + \gamma(\rho)\right\} \end{aligned} \tag{24}$$

From equ. (3) and (22) follows for the inner expansion evaluated at $X = -\infty$

$$\begin{aligned} t(\bar{x}) = T + \tau + \mu\Big\{\bar{x}\,\mathcal{U}_1^{-1} - A\mathcal{U}_1^{-1} &+ \bar{a}^{3/2}\log(-\bar{x}) \\ &+ \bar{a}^{3/2}\log\frac{2}{\mathcal{U}_1\bar{a}^{3/2}\bar{e}}\Big\} \end{aligned} \tag{25}$$

Now, if the phase angle T is chosen to be composed of several parts according to powers of μ as follows,

$$T = T_0 + \mu^{1/2} T_{1/2} + \mu T_1 \tag{26}$$

the third term in equ. (24) is to first order in μ, $-\frac{1}{2}\mu T_{\frac{1}{2}}^{2} U^{-1}$ and the two expressions for $t(\bar{x})$ can be made identical by making the following choices for T_0, U_1, and τ.

$$T_0 = t_{00}(1) \tag{27}$$

$$U_1 = U \tag{28}$$

$$\tau = -\mu^{1/2} T_{\frac{1}{2}} - \mu\Big\{ T_1 + \tfrac{1}{2} T_{\frac{1}{2}}^{2} U^{-1} - t_{01}(1) - (1+U^2)^{-3/2} \log \mu - \gamma(\rho) - A U_1^{-1} + \bar{a}^{3/2} \log \frac{2}{U_1 \bar{a}^{3/2} \bar{\varepsilon}} \Big\} \tag{29}$$

Note that this implies that $(1+U_1^{\,2})^{-1} = \bar{a}$; this will be confirmed by the matching of the expansions for y.

From equ. (8), (20) and (3) follow for the outer expansion of y evaluated near $x = 1$ and expressed in the inner variable $\bar{x}$,

$$y(\bar{x}) = \mu^{1/2} y_{1/2}(1,0) + \mu\left[(1+U^2)^{-3/2}\left(\log(-\bar{x}) + \log \mu\right) + \delta(\rho)\right] \tag{30}$$

Since the Keplerian part of this expression is multiplied with $\mu^{1/2}$, its value near $x = 1$ is obtained simply by substituting $x = 1$; no Taylor expansion need be used here because the second term would be proportional to $\mu^{3/2}$.

From equ. (3) and the expression for the inner expansion, equ. (21) follows for the inner expansion evaluated near $\bar{x} = -\infty$,

$$y(\bar{x}) = \mu\left[\frac{V_1}{\mathcal{U}_1}\bar{x} - K_1\right] + (t-T)$$

where it is again assumed that $(t-T)$ is small, so that $\sin(t-T) = (t-T)$.

This assumption is shown to be valid (at least to order μ) by equ. (25) and (29). If then the expression for $t(\bar{x})$ near $x=1$ (equ. (25)), and the evaluation of $\mathcal{T}$ (equ. (29)), which followed from the matching of the expressions for the time, are used, there follows for $y(\bar{x})$,

$$y(\bar{x}) = -\mu^{1/2}T_{\frac{1}{2}} + \mu\left[\frac{V_1}{\mathcal{U}_1}\bar{x} - K_1 + \bar{x}\mathcal{U}^{-1} + \bar{a}^{3/2}\log(-\bar{x})\right.$$
$$\left. + (1+\mathcal{U}^2)^{-3/2}\log\mu + t_{01}(1) - \tfrac{1}{2}T_{\frac{1}{2}}^2\mathcal{U}^{-1} + \gamma(\rho) - T_1\right] \tag{31}$$

The expressions (30) and (31) are made identical by the following choice of the constants $T_{\frac{1}{2}}$, V_1, K_1 and T_1.

$$T_{\frac{1}{2}} = -Y_{\frac{1}{2}}(1,0) \tag{32}$$

$$V_1 = -1 \tag{33}$$

$$-K_1 = \tfrac{1}{2}T_{\frac{1}{2}}^2\mathcal{U}^{-1} - t_{01}(1) + T_1 + \delta(\rho) - \gamma(\rho) \tag{34}$$

T_1 is arbitrary

The result $V_1 = -1$ confirms the expression $\bar{a} = (1+\mathcal{U}_1^2)^{-1}$ which was necessary for the time-matching since for the moon centered hyperbola $\bar{a} = \frac{1}{2\bar{h}} = (V_1^2 + \mathcal{U}_1^2)^{-1}$. With equs. (28), (33) and (34) the moon centered hyperbola is now determined, the constants $\bar{h}$, $\bar{\ell}$, $\bar{\rho}$ and $\bar{\varphi}$ being expressed as

$$
\begin{aligned}
2\bar{h} &= 1 + u_1^2 = 3 - 2\rho^2 \\
\bar{\ell} &= K_1 u_1 \\
\bar{p} &= K_1 u_1^2 + (1+u^2)^{-1/2} \\
\bar{q} &= K_1 u_1 - u_1 (1+u_1^2)^{-1/2}
\end{aligned}
\tag{35}
$$

These four constants are really equivalent to three integrals because

$$\bar{p}^2 + \bar{q}^2 = 2\bar{\ell}^2 \bar{h} + 1$$

so that a fourth integral is still needed. This is provided by the condition that $\bar{t} = 0$ at perilune. This condition is satisfied by the proper choice of the phase angle T and the origin of the inner variable $\bar{t}$ which are determined by equ. (27) for T_c, equ. (32) for $T_{\frac{1}{2}}$, and equ. (29) for τ. The constant A which is needed in equ. (29) is simply

$$A = -\frac{\bar{q}}{2\bar{h}}$$

It is a fortunate circumstance that T_1, the part of the phase angle T which is proportional to μ, is arbitrary. T_1 influences K_1, and thereby the angular momentum $\bar{\ell} = K_1 u_1$. With the hyperbola's total energy determined by u_1, the perilune distance can thus be adjusted by changing the angular momentum through T_1.

It may now be noted that the phase angle T (apart from the arbitrary contribution μT_1) and two of the hyperbolic constants depend only on the Keplerian part of the outbound trajectory. As a matter of fact, Lagerstrom and Kevorkian derived T_c and $T_{\frac{1}{2}}$ in the very beginning of their analysis and on the basis of the outbound Kepler trajectory alone. For the purpose of this presentation of their analysis it was felt that the modification in which $\mathcal{T}_0$ and $\mathcal{T}_{\frac{1}{2}}$ are derived from the matching conditions is a little more in line with the general principle of the method of singular perturbations; this principle being the determination of certain constants, which leave the inner and outer expansions indeterminate, from matching conditions.

Furthermore, it is noted that the first order corrections of the outbound trajectory enter into the matching conditions only through the functions $\gamma(\rho)$ (in the determination of τ) and $\left(\delta(\rho) - \gamma(\rho)\right)$ (in the determination of K_1). The functions γ and δ become unbounded as $\rho \longrightarrow 1$, i.e. for minimum energy trajectories, but the difference $(\delta - \gamma)$ was shown to be finite (ref. 11). The difference $[\delta - \gamma]$ may be interpreted as the correction of K_1, required if K_1 were determined on the basis of the Keplerian trajectory alone. Since K_1 is the $\bar{y}$-intercept of the approach asymptote of the moon-centered hyperbola at $X = 1$, it has been claimed that $(\delta - \gamma)$ is a measure of the error made in the usual methods of "patched-conic" computations; $(\delta - \gamma)$ is then simply the miss-distance of the approach trajectory. Because of the basis difference in the two methods (which has been pointed out earlier in this report: patching conics is geometric, while matching inner and outer expansions is analytic) a comparison on the basis of $(\delta - \gamma)$ tends to come out unfair for the patched-conic method. It would be interesting to see how the corrections t_1 and y_1 contribute to the outbound trajectory near its intersection with the moon's sphere of influence.

And if a thorough comparison of the two methods were desired, it should of course be based on final numerical results for representative trajectories computed by both methods. Lagerstrom and Kevorkian themselves have not provided such a comparison, except by pointing out that $(\mathcal{J} - Y)$ is a measure of the patched conic error; in ref. 5 there are comparisons with exact (i.e. numerically integrated) trajectories, but whether or not the results say much for the two-variable expansion method depends mostly on what kind of errors one is willing to except.

3.6 The Composite Solution

The outer and inner solutions have been formulated and their singular behavior has been identified. By matching these two solutions in their overlapping region of validity the phase angle and the constants of the moon centered hyperbola have been determined. To complete the work a composite solution must be formulated. According to the singular perturbation theory the composite solution is obtained by adding the outer and inner solutions and subtracting their common part. That common part is just the inner solution evaluated in the outer region (that is for at $\bar{X} \longrightarrow -\infty$), or the outer solution evaluated in the inner region (that is for $X \longrightarrow 1$); these two evaluations are identical because that was just the condition for matching. Here it is convenient to use the inner solution evaluated for $\bar{X} \longrightarrow -\infty$.

According to equ. (8) the outer solution is

$$Y(X, \mu) = \mu^{1/2} Y_{1/2}(X, \mu) + \mu Y_1(X)$$

where $Y_{1/2}$ and Y_1 are known functions, Y exhibiting a singularity for $X \to 1$.

According to equ. (3) the inner solution is

$$y(x,\mu) = \eta_m + \mu \bar{y}(\bar{x})$$

where the moon coordinate $\eta_m = \sin(t-T)$, $\bar{x} = \dfrac{x-\xi_m}{\mu}$ with $\xi_m = \cos(t-T)$ and $\bar{y}(\bar{x})$ is the equation of the moon-centered hyperbola. According to equ. (3) and (21), the inner solution evaluated for $\bar{x} \longrightarrow -\infty$ is

$$y(x,\mu)_{\bar{x}\to-\infty} = \eta_m + \alpha(\bar{x})$$

with $\bar{\alpha}(\bar{x}) = \dfrac{V_1}{u_1}\bar{x} - \dfrac{K_1}{u_1}$

The composite solution for y is thus

$$y(x,\mu) = \mu^{1/2} y_{\frac{1}{2}}(x,\mu) + \mu y_1(x) + \mu\left[\bar{y}(\bar{x}) - \alpha(\bar{x})\right]$$

and in the same way the composite solution for $t(x,\mu)$ is found to be

$$t(x,\mu) = t_o(x,\mu) + \mu t_1(x) + \mu\left[\bar{t}(\bar{x}) - \beta(\bar{x})\right] \tag{36}$$

where $$\beta(\bar{x}) = \frac{\bar{x}-A}{u_1} + \bar{a}^{3/2}\log\frac{-2\bar{x}}{u_1\,\bar{a}^{3/2}\bar{e}} \tag{37}$$

If analytical expressions for $t_1(x)$ and $y_1(x)$ were available it would be observed that their singularities are cancelled by the singularities of the expressions in square brackets; this is for instance the case in the analysis for the two-fixed center problem (ref. 3). In the absence of analytic expressions for t_1 and y_1, the singularities must cancel numerically. Now, to determine just the geometry of the moon centered hyperbola (determined by the constants in equ. (35), the functions t_1 and y_1 themselves are not needed, only the function $(\delta - \gamma)$ is. $(\delta - \gamma)$ depends on the initial condition $-\rho^2$ only and can be computed and tabulated once and for all. However, if the time-dependency and the entire trajectory is needed, the functions t_1 and y_1, as well as the expressions in the square brackets of equs. (36) and (37) must be computed and their singularities made to cancel numerically; this may be expected to cause some numerical difficulties.

4. THE OUTER EXPANSION IN ROTATING COORDINATES

In the previous section it was shown that the application of singular perturbation theory results in a uniformly valid solution to first order of for a certain class of trajectories in the restricted three body problem. In principle this is a satisfactory solution, but practically there are some difficulties because this solution is left in terms of quadratures which must be numerically integrated. Furthermore, since the formulation was carried out in a non-rotating coordinate system one may ask whether a formulation in a different coordinate system would be more advantageous.

Therefore, in conclusion, the following items are cited as possibly leading to improvements or analytical simplifications for this type of first order solution:

1) to obtain analytical approximations for the quadratures which depend on some parameter of the zero-order ellipse (in this case the energy);

2) to represent the problem in a rotating coordinate system as a third order system of differential equations by making use of the Jacobi Integral.

An investigation of the second recommendation has been initiated and in what follows the results for the outer solution are outlined in terms of quadratures. As a result of this investigation it was found that in addition to the choice of a rotating frame of reference the choice of polar coordinates was a decided advantage for the following reasons:

1) The solution for time is obtained from the first order differential equation provided by the Jacobi Integral;

2) The occurrence of elliptical integrals in the zero-order solution for the time is avoided when the radius is used as independent variable;

3) A solution in polar coordinates readily lends itself to extension to three dimensions.

The details of this analysis follow.

In the planar restricted three body problem assume a non-rotating earth-centered coordinate system with axes X, Y parallel to some inertial axes and let the earth-moon distance equal 1 while the masses of the earth and moon are $1-\mu$ and μ respectively and the gravitational constant $k^2=1$. The Lagrangian for a massless particle at (x, y) is from Reference 7:

$$L = \frac{1}{2}(\dot{x}^2 + \dot{y}^2) + \frac{1-\mu}{r} + \frac{\mu}{r_2} - \mu(x \cos t + y \sin t) \tag{38}$$

In this system the moon rotates with angular velocity $\omega = 1$ and the transformation to a rotating coordinate system X^*, Y^* with the moon at unit distance on the X^* axis is:

$$\begin{aligned} x^* &= x \cos t + y \sin t \\ y^* &= -x \sin t + y \cos t \end{aligned} \tag{39}$$

where in polar coordinates relative to the rotating coordinate system:

$$\begin{aligned} x^* &= r \cos \theta^* \\ y^* &= r \sin \theta^* \end{aligned} \tag{40}$$

and $r = r^*$. The Lagrangian in relative polar coordinates becomes:

$$L^* = \frac{1}{2}\left[\dot{r}^2 + (r\,\dot{\theta}^*)^2\right] + r^2\,\dot{\theta}^* + \frac{1-\mu}{r} + \frac{\mu}{r_2} - \mu\, r \cos\theta^* \tag{41}$$

where: $$r_2 = \left[1 + r^2 - 2r\cos\theta^*\right]^{1/2} = \left[(r-1)^2 - \frac{(y^*)^2}{r}\right]^{1/2}$$

Since L^* is time independent there exists an integral of the equations of motion known as the Jacobi integral which is equal to the Jacobi Constant C' for the relative energy. Thus the expression for the relative velocity becomes:

$$\dot{r}^2 + r^2 (\dot{\theta}^*)^2 = r^2 + \frac{2(1-\mu)}{r} + \frac{2\mu}{r_2} - 2\mu\ r \cos\theta^* - C' \tag{42}$$

An asymptotic series solution of the following form is to be obtained:

$$\begin{aligned} t &= t_o(r) + \mu\ t_1(r) + 0(\mu^2) \\ \theta^* &= \theta^*_o(r) + \mu\ \theta^*_1(r) + 0(\mu^2) \end{aligned} \tag{43}$$

where $\mu \sim .01$. The zero order solution is a two body ellipse relative to the earth with elements a and b, e, i, ω', Ω, τ and constant angular momentum ℓ_o and energy h_o. It will be assumed that the initial conditions are taken at the perigee. Then the solution for t is essentially a first order approximation to a "Kepler's Equation" for a special class of lunar trajectories in the planar restricted three body problem and t_o is exactly Kepler's equation for the two body problem:

$$t_o = \sqrt{\frac{a^3}{1-\mu}} \left(\cos^{-1}\left(\frac{a-r}{ae}\right) - e\sqrt{1 - \left(\frac{a-r}{ae}\right)^2} \right) \tag{44}$$

and θ^*_o is given by:

$$\theta^*_o = \cos^{-1}\left[\frac{a(1-e^2) - r}{re}\right] - t_o - \Psi_i$$

where Ψ_i is an initial phase angle between the semi major axis and the X^* axis. Such a zero order solution is valid since Lagerstrom and Kevorkian, Ref. 4, have shown that within a small neighborhood of the earth of $0(\mu^\alpha)$ the motion is Keplerian up to order $\mu^{1+3\alpha}$. Hereafter the subscript zero refers to values for the zero order solution and the subscript i refers to the initial conditions.

The Lagrange equation $\left[L\right]_\theta = 0$ provides the following expression for the change in the total angular momentum:

$$\frac{d}{dt}\left(r^2\dot{\theta}^* + r^2\right) = \mu\left[\frac{-r\sin\theta^*}{(r_2)^3} + r\sin\theta^*\right] \qquad (45)$$

Integrating for a first order approximation gives:

$$r^2\dot{\theta}^* + r^2 - \ell_o = \mu\int_{r_i}^{r}\left(\frac{-r\sin\theta o^*}{\left(r_{2(o)}\right)^3} + r\sin\theta_o^*\right)\frac{dt_o}{dr}\,dr \qquad (46)$$

Clearly the integrand in equ. (46) is expressible as a function of r through eqs. (44). However due to the transcendental nature of the resulting expression for the integrand an analytical integration cannot be obtained directly. Instead an approximation for the integral dependent on certain parameters of the zero order solution can be determined and exercising choice as to the form of the approximation will allow some simplification of the solution for t. Now $\dot{\theta}^*$ becomes:

$$\dot{\theta}_o^* + \mu\dot{\theta}_1^* = \frac{\ell_o}{r^2} - 1 + \mu P(r) \qquad (47)$$

where the approximation for the integral has been incorporated in P (r).

Now the Jacobi Integral, equ. (42) provides a first order differential equation for t after substituting for $\dot{\theta}^*$:

$$(t_o')^2 + 2\mu\, t_o'\, t_1' = \frac{r^2}{(2\,\ell_o - {}^2C)\, r^2 + 2(1-\mu)\, r - \ell_o^2 + 2\mu r^2 \left(\frac{1}{r_{2(0)}} + P(r)\,(r^2 - \ell_o) - \Delta - r\cos\theta_o^*\right)} \tag{48}$$

and

$$t_o' = \frac{r}{\sqrt{(2\,\ell_o - {}^2C)\, r^2 + 2(1-\mu)\, r - \ell_o^2}}$$

where ${}^2C = 2\,(\ell_o - h_o)$ is the energy constant for a two body orbit relative to a rotating reference frame and $\Delta = C' - {}^2C$. Note that in equ. (48) both $\frac{1}{r_2}$ and P(r) become unbounded as $r_2 \longrightarrow 0$; however, the combination of these terms should remain bounded insuring that $\frac{dr}{dt}$ is bounded near the moon.

Similarly $\theta^{*\prime}$ is obtained from equ. (48):

$$\theta_o^{*\prime} + \mu\theta_1^{*\prime} = \frac{\ell_o}{r^2}\, t_o' - t' + \mu\, \frac{\ell_o}{r^2}\, t_1' + \mu\, P(r)\, t_o' \tag{49}$$

where again the prime denotes differentiation with respect to r.

This completes the outline of the outer solution. A similar investigation of the inner solution and the results of matching the solutions will be final deciding factors in the determination of the practicality of this approach.

5. CONCLUSION

Interpreting the restricted three body problem as a singular perturbation problem results in a uniformly valid solution to first order in the small parameter μ for earth-moon trajectories. This solution can be thought of as being composed of an "outer solution," valid near the earth and an "inner solution," valid near the moon. The outer and inner solutions are matched in their common region of validity by determining certain constants (i.e. the initial phase angle of the moon and the elements of the moon-centered hyperbola) in such a way that the singularities which appear in the inner and outer solution vanish in the construction of the composite solution. The matching constants are expressed in terms of the initial conditions, with the exception of a part of order μ in the phase angle which can be chosen arbitrarily and can thus be used to adjust the lunar perigee distance.

It has been shown that the outer solution must necessarily contain a part that is proportional to the small parameter μ in order to make the match with the inner solution possible. A posteriori this conclusion could have been anticipated from a consideration of the order of magnitude of the angular momenta of inner and outer solutions. The need for this first order correction to the earth-centered outbound ellipse seems to explain why the usual patched conic methods (in which such a correction is not made) must be inaccurate. But such a statement must be made with some care, since in the two methods the matching is performed on a very different basis. In the two-variable expansion method the outer solution is evaluated at the moon's distance and equated to the inner solution evaluated far away from the moon, but far away in terms of the "blown-up" inner variable. Although this procedure makes good sense analytically, it is hard to see what it means geometrically. On the other hand, in the patched-conic method the earth centered ellipse (an uncorrected outer solution) is evaluated at the sphere of influence of the moon and equated to the moon centered hyperbola (the inner solution, but in physical variables) at that point. To make a sound comparison of the two methods, it should be based on the final numerical results, or at least one should determine how much the first order correction of the outer solution contributes to the Kepler ellipse up to the moon's sphere of influence.

The composite solutions, in particular the first order correction, is left in the form of quadratures for which no analytic expressions has been found yet. Therefore, although in theory the singularities of outer and inner so solutions cancel, the singularities must be evaluated numerically. This will cause numerical problems if the entire trajectory is to be known as a function of the time. On the other hand, if it is sufficient to just know the elements of the moon centered hyperbola, the quadratures need not be evaluated entirely. Only the parts of the first order correction indicated by $\gamma(\rho)$ and $\delta(\rho)$ are required, and in particular their difference $(\delta - \gamma)$. These functions depend only on the total energy $-\rho^2$ and can be evaluated once and for all for any interesting range of energies. There is an additional difficulty since δ and γ tend to infinity for minimum energy trajectories, but even there the difference $(\varsigma - \gamma)$ remains finite.

These difficulties may limit somewhat the practicality of the methods depending on how much trouble one would want to go through to write a computer program that evaluates the quadratures. Even so the method is of great interest and a similar development may be attempted along some different approach. Such a different approach is given in Section 4.

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REPUBLIC AVIATION CORPORATION

APPLICATION OF VARIATION OF PARAMETERS TO THE POLAR OBLATENESS PROBLEM

By

John Morrison
Henry Weinberg

Farmingdale, L. I., New York

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DEFINITION OF SYMBOLS

t	Time
f	True anomaly
$\underline{R}$	Position vector
r	$\lvert\underline{R}\rvert$ = magnitude of $\underline{R}$
μ	Gravitational constant
$\underline{G}$	Angular momentum vector
$\underline{P}$	Eccentricity vector
$\underline{Q}$	$\underline{G} \times \underline{P}$
$\underline{i}$	Unit vector in direction of x axis
$\underline{j}$	Unit vector in direction of y axis
$\underline{k}$	Unit vector in direction of z axis
e	Eccentricity
g	$\lvert\underline{G}\rvert$
p	$\lvert\underline{P}\rvert$
q	$\lvert\underline{Q}\rvert$
σ	Time of perigee passage
a	Semimajor axis
n	Mean motion
K_2	Coefficient of second harmonic of the potential due to the oblateness of the earth
A	$\dfrac{3\mu^2 K_2}{g^4}$

B	$\left(\frac{P_3}{p}\right)^2 + \left(\frac{Q_3}{q}\right)^2$
(r, θ)	Polar coordinate system introduced in x-y plane

SUBSCRIPTS

1,2,3	1st, 2nd, 3rd component of a vector
o	Initial value
s	Short periodic
ℓ	Long periodic

SUPERSCRIPTS

$\bullet$	Differentiation with respect to time
	Differentiation with respect to true anomaly

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SUMMARY

This report presents the derivation of a set of two body parameters and their associated perturbation equations. These equations are applied to the polar oblateness problem characterized by the second spherical harmonic. A modified Poisson method is used to obtain the first order solution to the problem. The modification of the method is introduced in order to eliminate the occurrence of secular terms which, because of the parameters employed, would have caused a rapid deterioration of the solution. The approximate solution is expressed as a function of true anomaly. Some analysis of second order theory is presented which suggests that difficulties with particular initial conditions may be avoided.

INTRODUCTION

Among the numerous troublesome aspects which one encounters in attempting to integrate the perturbation equations for the polar oblateness problem, two difficulties may occur which appear to be subject to, at least some amelioration.

In general, there are two decisions one must make before these difficulties become apparent. These decisions consist of selecting a set of parameters and a method of integrating the perturbation equations. The possible sets of two-body parameters may be divided into two groups, one of which contains canonical parameters and one which does not. Two methods of integration, in general use, are Poisson's method (1) and Von Zeipel's method (2). The latter method is applied only to canonical parameters. In most instances, regardless of the set of two-body parameters or method of integration employed, the results present two interesting properties. The first is the occurrence of terms in the approximate solution which show a secular growth. The second is the presence of singularities in the second order corrections for certain initial conditions of the parameters. The first property is not, in general, objectionable since the secular terms usually appear in the expressions for angle parameters. However, for some parameters, such as the unit perigee vector, the occurrence of secular terms destroys the unit characteristic and limits the applicability of the results to relatively short time intervals.

It is proposed in this report to derive a set of parameters and their associated perturbation equations which, when applied to the polar oblateness problem, yield, after approximate integration, equations for the parameters which manifest no secular growth to the first order, except for one element. A brief analysis of the structure of the second order perturbation equations is developed which suggests that the occurrence of singularities arising from initial conditions is not a necessary concomitant of the polar oblateness problem. The application of second order theory, however, will not be attempted in this report, because the parameters which have been chosen degenerate for nearly circular orbits. Even though the set of parameters employed is defective, the comparative simplicity of the perturbation equations recommends the use of these parameters for a clearer insight into the particular difficulties which their use is intended to eliminate. It should be noted that the degeneracy of the parameters for nearly circular orbits is not a case of replacing one difficulty with another, but is simply a consequence of the choice of parameters and not of the integration technique. A more judicious choice of parameters has been made and an improved integration technique developed which eliminates the imperfections in the present method. A report is now in preparation which incorporates these developments.

DERIVATION OF A SET OF PARAMETERS FOR THE KEPLER PROBLEM

To specify the solution of the vector equation

$$\ddot{\underline{R}} + \frac{\mu \underline{R}}{r^3} = 0 \tag{1}$$

six independent parameters are needed. For the purposes of this report, the following set will be used:

- σ, the time of perigee passage;
- $\underline{P}$, the eccentricity vector;
- $\underline{Q}$, a vector perpendicular to $\underline{P}$ and lying in the plane of motion.

At first glance it would appear that this set contains seven independent elements; but, since $\underline{P}$ and $\underline{Q}$ are mutually orthogonal, any one component may be expressed as a function of the remaining five. The vectors $\underline{P}$ and $\underline{Q}$ may be obtained from Eq. (1) in the following manner: Take the cross product of $\underline{R}$ and Eq. (1)

$$\underline{R} \times \ddot{\underline{R}} = 0 \tag{2}$$

Integration of Eq. (2) gives

$$\underline{R} \times \dot{\underline{R}} = \underline{G} \tag{3}$$

in which $\underline{G}$ is the constant angular momentum vector. Now take the cross product of Eq. (1) and $\underline{G}$

$$\ddot{\underline{R}} \times \underline{G} + \frac{\mu \underline{R}}{r^3} \times \underline{G} = 0 \tag{4}$$

After expanding $\underline{R} \times \underline{G}$ using Eq. (3) and recalling that $\underline{G}$ is constant, Eq. (4) integrates to

$$\dot{\underline{R}} \times \underline{G} - \frac{\mu \underline{R}}{r} = \underline{P} \tag{5}$$

in which $\underline{P}$ is a constant vector. To find the magnitude and direction of $\underline{P}$ rewrite Eq. (5) in the form

$$\underline{P} = \underline{R}\left(\dot{\underline{R}} \cdot \dot{\underline{R}} - \frac{\mu}{r}\right) - \dot{\underline{R}}\,(\underline{R} \cdot \dot{\underline{R}}) \tag{6}$$

Evaluating Eq. (6) at perigee yields

$$\underline{P} = \underline{U}_p\,\mu e \tag{7}$$

where

e is the eccentricity of the orbit

and $\underline{U}_p$ is a unit vector in the direction of perigee. Let $\underline{Q}$ be defined by

$$\underline{Q} = \underline{G} \times \underline{P} = \frac{\mu}{r}\,\underline{R} \times \underline{G} + \dot{\underline{R}}\,g^2 \tag{8}$$

The magnitudes of $\underline{G}$, $\underline{P}$, and $\underline{Q}$ are g, $p = \mu e$, and $q = gp$, respectively.

Since $\underline{R}$, $\underline{P}$, and $\underline{Q}$ are coplanar, $\underline{R}$ may be expressed as a linear combination of $\underline{P}$ and $\underline{Q}$

$$\underline{R} = \alpha_1\,\underline{P} + \alpha_2\,\underline{Q} \tag{9}$$

The scalar product of Eq. (9) with $\underline{P}$ yields

$$\alpha_1 = \frac{\underline{R} \cdot \underline{P}}{p^2} = \frac{r \cos f}{p} \tag{10}$$

where f is the true anomaly of $\underline{R}$. Similarly,

$$\alpha_2 = \frac{\underline{R} \cdot \underline{Q}}{q^2} = \frac{r \sin f}{q} \tag{11}$$

$\dot{\underline{R}}$ may be written as

$$\dot{\underline{R}} = \dot{\alpha}_1\,\underline{P} + \dot{\alpha}_2\,\underline{Q} \tag{12}$$

Making use of the well known formulas

$$r = \frac{g^2}{\mu\,(1 + e \cos f)} \tag{13}$$

$$\dot{f} = \frac{g}{r^2} \tag{14}$$

it follows that

$$\dot{\alpha}_1 = -\frac{\mu \sin f}{gp} \tag{15}$$

$$\dot{\alpha}_2 = \frac{e + \cos f}{gq} \mu \tag{16}$$

PERTURBATION EQUATIONS

After having obtained a set of parameters the first step in deriving the perturbation equations is to introduce the perturbing force $\underline{F}$ on the R. H. S. of Eq. (1) which gives

$$\ddot{\underline{R}} + \frac{\mu \underline{R}}{r^3} = \underline{F} \tag{17}$$

The perturbing force $\underline{F}$ will cause $\underline{R}$ to deviate from the Keplerian orbit, and a new solution must be found. This solution can also be put in the form of Eq. (9), but now the parameters $\underline{G}$, $\underline{P}$ and $\underline{Q}$ will be functions of time. In order to determine the time dependences, it will be necessary to obtain the differential equations for the parameters in so far as they depend on the perturbing force $\underline{F}$.

Differentiation of Eq. (3) gives

$$\dot{\underline{G}} = \underline{R} \times \ddot{\underline{R}} \tag{18}$$

Substitution of Eq. (17) yields

$$\dot{\underline{G}} = \underline{R} \times \underline{F} \tag{19}$$

Similarly, differentiation of Eq. (5) gives

$$\dot{\underline{P}} = \ddot{\underline{R}} \times \underline{G} + \dot{\underline{R}} \times \dot{\underline{G}} + \mu \frac{\underline{\underline{R \times G}}}{r^3} \tag{20}$$

Substituting for $\dot{\underline{G}}$ and $\ddot{\underline{R}}$ yields

$$\dot{\underline{P}} = \underline{F} \times \underline{G} + \dot{\underline{R}} \times (\underline{R} \times \underline{F}) \tag{21}$$

From Eqs. (8), (19) and (21), $\dot{\underline{Q}}$ is given by

$$\dot{\underline{Q}} = \frac{\mu}{r} \underline{R} \times (\underline{R} \times \underline{F}) + \underline{F}\, g^2 + 2\, \dot{\underline{R}}\, (\underline{G} \cdot \dot{\underline{G}}) \tag{22}$$

The equation for the variation of σ, the time of perigee passage, is derived from Kepler's equation, which, for $0 < e < 1$, takes the form

$$n\,(t - \sigma) = \tan^{-1} \sin f \frac{\sqrt{1 - e^2}}{e + \cos f} - \sin f\, \frac{e\sqrt{1 - e^2}}{1 + e \cos f} \tag{23}$$

where $n = \sqrt{\frac{\mu}{+a^3}}$ and $g = \sqrt{+\,\mu a\,(1-e^2)}$.

For $e > 1$, Kepler's equation is given by

$$n\,(t - \sigma) = \tanh^{-1} \sin f \frac{\sqrt{e^2 - 1}}{e + \cos f} - \sin f\, \frac{e\sqrt{e^2 - 1}}{1 + e \cos f} \tag{23'}$$

where $n = \sqrt{\frac{\mu}{-a^3}}$ and $g = \sqrt{-\mu a(e^2 - 1)}$. Using various identities, Eqs. (23) may be put in the following form

$$n\,(t - \sigma) = \tan^{-1} \frac{\underline{R} \cdot \dot{\underline{R}}}{(1 - \frac{r}{a})\, a^2 n} - \frac{\underline{R} \cdot \dot{\underline{R}}}{a^2\, n} \tag{24}$$

$$n\,(t - \sigma) = \tanh^{-1} \frac{\underline{R} \cdot \dot{\underline{R}}}{(1 - \frac{r}{a})\, a^2 n} - \frac{\underline{R} \cdot \dot{\underline{R}}}{a^2\, n} \tag{24'}$$

Differentiation of these equations with respect to time, and substitution of Eq. (17) for $\ddot{\underline{R}}$ gives, in either case

$$\dot{\sigma} = \underline{F} \cdot \left\{ -\frac{3a}{\mu} \dot{\underline{R}}\,(t - \sigma) + \frac{a}{\mu} \underline{R} + \frac{a^2}{p^2} \left[(1 - e^2)\, (\dot{\underline{R}} \cdot \underline{R}) \dot{\underline{R}} - \frac{r}{a} \underline{P} \right] \right\} \tag{25}$$

where

$$-\frac{1}{a} = \frac{\dot{\underline{R}} \cdot \dot{\underline{R}}}{\mu} - \frac{2}{r}$$

and

$$\dot{a} = \dot{\underline{R}} \cdot \underline{F} \left(\frac{2a^2}{\mu}\right)$$

It is convenient to have available the total time derivative of true anomaly. Differentiating the expression

$$\cos f = \frac{\underline{\underline{R}}}{r} \cdot \frac{\underline{\underline{P}}}{p} \tag{26}$$

it follows that

$$-(\sin f)\, \dot{f} = \frac{r\, \dot{\underline{R}} - \dot{\underline{R}}\, \dot{r}}{r^2} \cdot \frac{\underline{\underline{P}}}{p} + \frac{\underline{\underline{R}}}{r} \cdot \left(\frac{\underline{\underline{P}}}{p}\right) \tag{27}$$

and therefore

$$\dot{f} = \frac{g}{r^2} - \frac{\underline{\underline{P}}}{p} \cdot \frac{\underline{Q}}{q} \tag{28}$$

APPLICATION OF THE PERTURBATION EQUATIONS TO THE POLAR OBLATENESS PROBLEM

In this report, the polar oblateness problem will be assumed to be characterized by the perturbing potential

$$\Phi = \frac{\mu K_2}{r^3} \left(1 - 3\,\frac{z^2}{r^2}\right) \tag{29}$$

In order to apply the perturbation equations, previously presented, to this problem, it is necessary to specify the perturbing force $\underline{F}$. This force is the gradient of the perturbing potential Φ.

$$\underline{F} = -\frac{3\mu K_2}{r^5}\left\{\left[1 - 5\frac{z^2}{r^2}\right]\underline{R} + 2z\,\underline{k}\right\} \tag{30}$$

The procedure for applying the perturbation equations may be outlined as follows:

(a) Reexpress the perturbation equations in terms of the parameters $\underline{P}$, $\underline{Q}$, and $\underline{G}$, and true anomaly, f, by substituting Eqs. (9), (12), and (30) for $\underline{R}$, $\dot{\underline{R}}$, and $\underline{F}$, respectively.

(b) Since the resulting equations are functions of true anomaly, it is legitimate to take $\dot{f} = g/r^2$, for a first order approximation. It follows that the differential equations with respect to time may be transformed to differential equations with respect to true anomaly.

(c) These perturbation equations are now written as Fourier polynomials. Terms with constant coefficients are transposed to the L. H. S.

(d) To obtain a first order solution for the system of equations derived in (c), all parameters on the R. H. S. and the parameter g, wherever it occurs, are held constant. Under these conditions, the system can be solved exactly.

(e) The perturbation equation for the parameter σ is treated similarly with some modifications.

Carrying out the operations indicated in (a), (b), and (c) the results are:

$$\underline{P}' - \frac{3\mu^3 K_2}{g^4}\, e\left\{\underline{k}\,\frac{Q_3}{q} - \frac{\underline{P}}{p}\,\frac{P_3 Q_3}{p\,q} - \frac{\underline{Q}}{q}\left[\frac{3}{2}\,\frac{P_3^2}{p^2} + \frac{5}{2}\,\frac{Q_3^2}{q^2} - 1\right]\right\}$$

$$= \frac{3\mu K_2}{g^4}\left\{\underline{k}\left[\frac{P_3}{p}\left(\frac{e^2}{2}\sin 3f + e\sin 2f + \frac{e^2}{2}\sin f\right) - \frac{Q_3}{q}\left(\frac{e^2}{2}\cos 3f\right.\right.\right.$$

$$\left.\left. + e\cos 2f - \frac{e^2}{2}\cos f\right)\right] + \frac{\underline{P}}{p}\left[\left(\frac{P_3}{p}\right)^2\left(\frac{5e^2}{16}\sin 5f + \frac{3e}{2}\sin 4f\right. \tag{31}$$

$$\left. + \left(\frac{7}{4} + \frac{15e^2}{16}\right)\sin 3f + 3e\sin 2f + \left(\frac{7}{4} + \frac{5e^2}{8}\right)\sin f\right) - \frac{P_3 Q_3}{p\,q}\left(\frac{5e^2}{8}\cos 5f\right.$$

$$\left. + 3e\cos 4f + \left(\frac{7}{2} + \frac{13}{8}e^2\right)\cos 3f + 4e\cos 2f + \left(\frac{1}{2} + \frac{7}{4}e^2\right)\cos f\right)$$

$$-\left(\frac{Q_3}{q}\right)^2\left(\frac{5e^2}{16}\sin 5f+\frac{3e}{2}\sin 4f+\left(\frac{7}{4}+\frac{11e^2}{16}\right)\sin 3f+e\sin 2f\right.$$

$$\left.+\left(\frac{3e^2}{8}-\frac{5}{4}\right)\sin f\right)-\left(\frac{e^2}{4}\sin 3f+e\sin 2f+\left(1+\frac{e^2}{4}\right)\sin f\right)\Bigg]$$

$$+\frac{Q}{q}\Bigg[-\left(\frac{P_3}{p}\right)^2\left(\frac{5e^2}{16}\cos 5f+\frac{3e}{2}\cos 4f+\left(\frac{7}{4}+\frac{17}{16}e^2\right)\cos 3f+3e\cos 2f\right.$$

$$\left.+\left(\frac{5}{4}+\frac{13}{8}e^2\right)\cos f\right)+\left(\frac{Q_3}{q}\right)^2\left(\frac{5e^2}{16}\cos 5f+\frac{3e}{2}\cos 4f+\left(\frac{7}{4}+\frac{13}{16}e^2\right)\cos 3f\right.$$

$$\left.+e\cos 2f-\left(\frac{7}{4}+\frac{9e^2}{8}\cos f\right)\right)-\frac{P_3Q_3}{p\,q}\left(\frac{5e^2}{2}\sin 5f+3e\sin 4f+\left(\frac{7}{2}+3e^2\right)\sin 3f\right.$$

$$\left.+4e\sin 2f+\left(\frac{e^2}{2}-\frac{1}{2}\right)\sin f\right)+\left(\frac{e^2}{4}\cos 3f+e\cos 2f+\left(1+\frac{3e^2}{4}\right)\cos f\right)\Bigg]\Bigg\}$$

(31) cont'd

$$\underline{Q}'-\frac{3\mu^3K_2e}{g^3}\left\{-\underline{k}\frac{P_3}{p}+\frac{P}{p}\left[\frac{5}{2}\frac{P_3^{\,2}}{p^2}+\frac{3}{2}\frac{Q_3^{\,2}}{q^2}-1\right]+\frac{Q}{q}\frac{P_3}{p}\frac{Q_3}{q}\right\}$$

$$=\frac{3\mu^3K_2}{g^3}\left\{-\underline{k}\left[\frac{P_3}{p}\left(\frac{e^2}{2}\cos 3f+e\cos 2f+\frac{3e^2}{2}\cos f\right)+\frac{Q_3}{q}\left(\frac{e^2}{2}\sin 3f\right.\right.\right.$$

$$\left.\left.+e\sin 2f+\frac{e^2}{2}\sin f\right)\right]+\frac{P}{p}\left[\left(\frac{P_3}{p}\right)^2\left(\frac{5e^2}{16}\cos 5f+\frac{3e}{2}\cos 4f\right.\right.$$

$$\left.+\left(\frac{7}{4}+\frac{25}{16}e^2\right)\cos 3f+4e\cos 2f+\left(\frac{5}{4}+\frac{25}{8}e^2\right)\cos f\right)-\left(\frac{Q_3}{q}\right)^2\left(\frac{5e^2}{16}\cos 5f\right.$$

$$\left.+\frac{3e}{2}\cos 4f+\left(\frac{7}{4}+\frac{5e^2}{16}\right)\cos 3f-\left(\frac{7}{4}+\frac{5}{8}e^2\right)\cos f\right)+\frac{P_3Q_3}{p\,q}\left(\frac{5e^2}{2}\sin 5f\right.$$

$$+ 3e \sin 4f + (\frac{7}{2} + 3e^2) \sin 3f + 4e \sin 2f + (\frac{e^2}{2} - \frac{1}{2}) \sin f\Big) - \Big(\frac{e^2}{4} \cos 3f$$

$$+ e \cos 2f + (1 + \frac{3e^2}{4}) \cos f\Big)\Big] + \frac{Q}{q}\Big[\Big(\frac{P_3}{p}\Big)^2 \Big(\frac{5e^2}{16} \sin 5f + \frac{3e}{2} \sin 4f$$

$$+ (\frac{7}{4} + \frac{31}{16} e^2) \sin 3f + 5e \sin 2f + (\frac{7}{4} + \frac{13e^2}{8}) \sin f\Big) - \Big(\frac{Q_3}{q}\Big)^2 \Big(\frac{5e^2}{16} \sin 5f$$

$$+ \frac{3e}{2} \sin 4f + (\frac{7}{4} + \frac{11e^2}{16}) \sin 3f + e \sin 2f + (\frac{3e^2}{8} - \frac{5}{4}) \sin f\Big)$$

$$- \frac{P_3 Q_3}{p\ q}\Big(\frac{5e^2}{8} \cos 5f + 3e \cos 4f + (\frac{7}{2} + \frac{21}{8} e^2) \cos 3f + 6e \cos 2f$$

$$+ (\frac{1}{2} + \frac{3e^2}{4}) \cos f\Big) - \Big(\frac{e^2}{4} \sin 3f + e \sin 2f + (1 + \frac{3e^2}{4}) \sin f\Big)\Big]\Big\} \qquad (31)\text{ cont'd}$$

$$\underline{G}' + \frac{3\mu^2 K_2}{g^3} \Big\{\frac{P}{p} \frac{P_3}{p} + \frac{Q}{q} \frac{Q_3}{q}\Big\} \times \underline{k}$$

$$= \frac{3\mu^2 K_2}{g^3} \Big\{\frac{P}{p}\Big[\frac{P_3}{p}\Big(\frac{e}{2} \cos 3f + \cos 2f + \frac{3e}{2} \cos f\Big) + \frac{Q_3}{q}\Big(\frac{e}{2} \sin 3f + \sin 2f$$

$$+ \frac{e}{2} \sin f\Big)\Big] + \frac{Q}{q}\Big[\frac{P_3}{p}\Big(\frac{e}{2} \sin 3f + \sin 2f + \frac{e}{2} \sin f\Big) + \frac{Q_3}{q}\Big(- \frac{e}{2} \cos 3f - \cos 2f$$

$$+ \frac{e}{2} \cos f\Big)\Big]\Big\} \times \underline{k}$$

where

$$(\quad)' = \frac{d(\quad)}{df}$$

Consider the system of homogeneous equations obtained by setting the R. H. S. of Eqs. (31) equal to zero.

$$\underline{P}'_\ell - \frac{3\mu^2 K_2 p_\ell}{g_\ell^4}\left\{\underline{k}\,\frac{Q_{3\ell}}{q_\ell} - \frac{\underline{P}_\ell}{p_\ell}\,\frac{P_{3\ell}}{p_\ell}\,\frac{Q_{3\ell}}{q_\ell} - \frac{\underline{Q}_\ell}{q_\ell}\left[\frac{3}{2}\,\frac{P_{3\ell}^2}{p_\ell^2} + \frac{5}{2}\,Q_{3\ell}^2 - 1\right]\right\} = 0$$

$$\underline{Q}'_\ell - \frac{3\mu^2 K_2 q_\ell}{g_\ell^4}\left\{-\underline{k}\,\frac{P_{3\ell}}{p_\ell} + \frac{\underline{P}_\ell}{p_\ell}\left[\frac{5}{2}\,\frac{P_{3\ell}^2}{p_\ell^2} + \frac{3}{2}\,\frac{Q_{3\ell}^2}{q_\ell^2} - 1\right] + \frac{\underline{Q}_\ell}{q_\ell}\,\frac{P_{3\ell}}{p_\ell}\,\frac{Q_{3\ell}}{q_\ell}\right\} = 0 \tag{32}$$

$$\underline{G}'_\ell + \frac{3\mu^2 K_2 g_\ell}{g_\ell^4}\left\{\frac{\underline{P}_\ell}{p_\ell}\,\frac{P_{3\ell}}{p_\ell} + \frac{\underline{Q}_\ell}{q_\ell}\,\frac{Q_{3\ell}}{q_\ell}\right\} \times \underline{k} = 0$$

It will become apparent that $\underline{P}_\ell$, $\underline{Q}_\ell$, and $\underline{G}_\ell$ represent the long periodic terms of $\underline{P}$, $\underline{Q}$, and $\underline{G}$, respectively.

For this system of equations, Eq. (8), $\underline{Q}_\ell = \underline{G}_\ell \times \underline{P}_\ell$ still holds. Since

$$p^2_\ell = \underline{P}_\ell \cdot \underline{P}_\ell \qquad q^2_\ell = \underline{Q}_\ell \cdot \underline{Q}_\ell \qquad g^2_\ell = \underline{G}_\ell \cdot \underline{G}_\ell$$

It follows from Eqs. (32) that

$$p'_\ell = \frac{\underline{P}_\ell}{p_\ell} \cdot \underline{P}'_\ell = 0$$

$$q'_\ell = \frac{\underline{Q}_\ell}{q_\ell} \cdot \underline{Q}'_\ell = 0 \tag{33}$$

$$g'_\ell = \frac{\underline{G}_\ell}{g_\ell} \cdot \underline{G}'_\ell = 0$$

Therefore, for this system of equations, p_ℓ, q_ℓ, and g_ℓ are constant. Similarly,

$$\frac{P_{3\ell}}{p_\ell}\left(\frac{P_{3\ell}}{p_\ell}\right)' + \frac{Q_{3\ell}}{q_\ell}\left(\frac{Q_{3\ell}}{q_\ell}\right)' = 0 \tag{34}$$

so that

$$\left(\frac{P_{3\ell}}{p_\ell}\right)^2 + \left(\frac{Q_{3\ell}}{q_\ell}\right)^2 = B \tag{35}$$

is constant.

Using the identity

$$\underline{k} \times \frac{\underline{G}_\ell}{g_\ell} = \frac{\underline{P}_\ell}{p_\ell}\frac{Q_{3\ell}}{q_\ell} - \frac{\underline{Q}_\ell}{q_\ell}\frac{P_{3\ell}}{p_\ell} \tag{36}$$

it follows that

$$\begin{aligned}\left(\frac{\underline{P}_\ell}{p_\ell}\frac{P_{3\ell}}{p_\ell} + \frac{\underline{Q}_\ell}{q_\ell}\frac{Q_{3\ell}}{q_\ell}\right) \times \underline{k} &= \left(\frac{\underline{Q}_\ell}{q_\ell} \times \frac{\underline{G}_\ell}{g_\ell}\frac{P_{3\ell}}{p_\ell} + \frac{\underline{G}_\ell}{g_\ell} \times \frac{\underline{P}_\ell}{p_\ell}\frac{Q_{3\ell}}{q_\ell}\right) \times \underline{k} \\ &= \frac{G_{3\ell}}{g_\ell}\left[\frac{\underline{P}_\ell}{p_\ell}\frac{Q_{3\ell}}{q_\ell} - \frac{\underline{Q}_\ell}{q_\ell}\frac{P_{3\ell}}{p_\ell}\right] = \frac{G_{3\ell}}{g_\ell}\underline{k} \times \frac{\underline{G}_\ell}{g_\ell}\end{aligned} \tag{37}$$

Therefore, Eqs. (32), can be rewritten as

$$\begin{aligned}&\underline{P}'_\ell + Ap_\ell\left\{\frac{P_{3\ell}}{p_\ell}\underline{k} \times \frac{\underline{G}_\ell}{g_\ell} - \frac{\underline{Q}_\ell}{q_\ell}\left(1 - \frac{5}{2}B\right) - \underline{k}\frac{Q_{3\ell}}{q_\ell}\right\} = 0 \\ &\underline{Q}'_\ell + Aq_\ell\left\{\frac{Q_{3\ell}}{q_\ell}\underline{k} \times \frac{\underline{G}_\ell}{g_\ell} - \frac{\underline{P}_\ell}{p_\ell}\left(\frac{5}{2}B - 1\right) + \underline{k}\frac{P_{3\ell}}{p_\ell}\right\} = 0 \\ &\underline{G}'_\ell + Ag_\ell\left\{\frac{G_{3\ell}}{g_\ell}\underline{k} \times \frac{\underline{G}_\ell}{g_\ell}\right\} = 0\end{aligned} \tag{38}$$

where $A = \dfrac{3\mu^2 K_2}{g_\ell^4}$

The third components of $\underline{P}'_\ell$, $\underline{Q}'_\ell$, and $\underline{G}'_\ell$ are

$$P'_{3\ell} + Ap_\ell \frac{Q_{3\ell}}{q_\ell}\left(\frac{5}{2}B - 2\right) = 0$$

$$Q'_{3\ell} + Aq_\ell \frac{P_{3\ell}}{p_\ell}\left(2 - \frac{5}{2}B\right) = 0 \tag{39}$$

$$G'_{3\ell} = 0$$

which form a system of first order, linear, homogeneous differential equations with constant coefficients. The solution is

$$P_{3\ell} = P_{3o} \cos\left\{A\left(\frac{5}{2}B - 2\right)f\right\} - Q_{3o} \sin\left\{A\left(\frac{5}{2}B - 2\right)f\right\}$$

$$Q_{3\ell} = P_{3o} \sin\left\{A\left(\frac{5}{2}B - 2\right)f\right\} + Q_{3o} \cos\left\{A\left(\frac{5}{2}B - 2\right)f\right\} \tag{40}$$

$$G_{3\ell} = G_{3o}$$

where P_{3o}, Q_{3o}, and G_{3o} are initial conditions. Similarly, the first two components of $\underline{G}'$ are

$$G'_{1\ell} - A\, G_{2\ell} \frac{G_{3\ell}}{g_\ell} = 0$$

$$G'_{2\ell} + A\, G_{1\ell} \frac{G_{3\ell}}{g_\ell} = 0 \tag{41}$$

This system has the solution

$$G_{1\ell} = G_{1o} \cos\left(A \frac{G_{3\ell}}{g_\ell} f\right) + G_{2o} \sin\left(A \frac{G_{3\ell}}{g_\ell} f\right)$$

$$G_{2\ell} = -\, G_{1o} \sin\left(A \frac{G_{3\ell}}{g_\ell} f\right) + G_{2o} \cos\left(A \frac{G_{3\ell}}{g_\ell} f\right) \tag{42}$$

where G_{1o} and G_{2o} are initial conditions. Using the identities,

$$\frac{G_{1\ell}}{g_\ell}\frac{G_{3\ell}}{g_\ell} = -\frac{P_{1\ell}}{p_\ell}\frac{P_{3\ell}}{p_\ell} - \frac{Q_{1\ell}}{q_\ell}\frac{Q_{3\ell}}{q_\ell}$$

$$\frac{G_{2\ell}}{g_\ell}\frac{G_{3\ell}}{g_\ell} = -\frac{P_{2\ell}}{p_\ell}\frac{P_{3\ell}}{p_\ell} - \frac{Q_{2\ell}}{q_\ell}\frac{Q_{3\ell}}{q_\ell} \tag{43}$$

Eqs. (41) may be transformed into

$$G'_{1\ell} + A\, g_\ell \left\{\frac{P_{2\ell}}{p_\ell}\frac{P_{3\ell}}{p_\ell} + \frac{Q_{2\ell}}{q_\ell}\frac{Q_{3\ell}}{q_\ell}\right\} = 0$$

$$G'_{2\ell} - A\, g_\ell \left\{\frac{P_{1\ell}}{p_\ell}\frac{P_{3\ell}}{p_\ell} + \frac{Q_{1\ell}}{q_\ell}\frac{Q_{3\ell}}{q_\ell}\right\} = 0 \tag{44}$$

Eqs. (44) together with the identities

$$\frac{G_{1\ell}}{g_\ell} = \frac{P_{2\ell}}{p_\ell}\frac{Q_{3\ell}}{q_\ell} - \frac{P_{3\ell}}{p_\ell}\frac{Q_{3\ell}}{q_\ell}$$

$$\frac{G_{2\ell}}{g_\ell} = \frac{P_{3\ell}}{p_\ell}\frac{Q_{1\ell}}{q_\ell} - \frac{P_{1\ell}}{p_\ell}\frac{Q_{3\ell}}{q_\ell} \tag{45}$$

determine the remaining components of $\underline{P}$ and $\underline{Q}$ which are

$$P_{1\ell} = \frac{p_\ell}{AB}\left\{\frac{P_{3\ell}}{p_\ell}\left(\frac{G_{2\ell}}{g_\ell}\right)' - A\frac{G_{2\ell}}{g_\ell}\frac{Q_{3\ell}}{q_\ell}\right\}$$

$$P_{2\ell} = \frac{p_\ell}{AB}\left\{A\frac{G_{1\ell}}{g_\ell}\frac{Q_{3\ell}}{q_\ell} - \frac{P_{3\ell}}{p_\ell}\left(\frac{G_{1\ell}}{g_\ell}\right)'\right\}$$

$$Q_{1\ell} = \frac{q_\ell}{AB}\left\{A\frac{G_{2\ell}}{g_\ell}\frac{P_{3\ell}}{p_\ell} + \frac{Q_{3\ell}}{q_\ell}\left(\frac{G_{2\ell}}{g_\ell}\right)'\right\} \tag{46}$$

$$Q_{2\ell} = \frac{q_\ell}{AB}\left\{-A\frac{G_{1\ell}}{g_\ell}\frac{P_{3\ell}}{p_\ell} - \frac{Q_{3\ell}}{q_\ell}\left(\frac{G_{1\ell}}{g_\ell}\right)'\right\}$$

All quantities appearing on the R.H.S. of Eqs. (46) are known. After some algebraic manipulation, the solution for the system of Eqs. (32) may be expressed as

$$\begin{bmatrix} P_1 \\ P_2 \\ P_3 \\ Q_1 \\ Q_2 \\ Q_3 \\ G_1 \\ G_2 \\ G_3 \end{bmatrix} = \begin{bmatrix} C & 0 & 0 \\ 0 & C & 0 \\ 0 & 0 & C \end{bmatrix} \cdot \begin{bmatrix} \cos A\left(\frac{5}{2}B-2\right) f\, I & -\sin A\left(\frac{5}{2}B-2\right) f\, I & 0 \\ \sin A\left(\frac{5}{2}B-2\right) f\, I & \cos A\left(\frac{5}{2}B-2\right) f\, I & 0 \\ 0 & 0 & I \end{bmatrix} \begin{bmatrix} P_{1o} \\ P_{2o} \\ P_{3o} \\ Q_{1o} \\ Q_{2o} \\ Q_{3o} \\ G_{1o} \\ G_{2o} \\ G_{3o} \end{bmatrix} \quad (47)$$

where

$$C = \begin{bmatrix} \cos A \frac{G_{3\ell}}{g_\ell} f & \sin A \frac{G_{3\ell}}{g_\ell} f & 0 \\ -\sin A \frac{G_{3\ell}}{g_\ell} f & \cos A \frac{G_{3\ell}}{g_\ell} f & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

and

$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

To find the particular solution of Eqs. (31), assume a solution of the form (47) where $\underline{P}_o$, $\underline{Q}_o$, and $\underline{G}_o$ are functions of f. Substituting solution (47) into the L.H.S. of Eqs. (31) will yield three equations for $\underline{P}_o{}'$, $\underline{Q}_o{}'$, and $\underline{G}_o{}'$. After solving for these derivatives, and recalling condition (d), $\underline{P}_o$, $\underline{Q}_o$, and $\underline{G}_o$ may then be found by integration alone. If the second order terms in this solution are neglected, the results are equivalent to integrating the R.H.S. of Eqs. (31) and adding the results to solution (47). The first order solution for $\underline{P}$, $\underline{Q}$, $\underline{G}$, is

$$\underline{P} = \underline{P}_{\ell} + \left[\underline{P}_s\right]_{f_o}^{f}$$

$$\underline{Q} = \underline{Q}_{\ell} + \left[\underline{Q}_s\right]_{f_o}^{f} \tag{48}$$

$$\underline{G} = \underline{G}_{\ell} + \left[\underline{G}_s\right]_{f_o}^{f}$$

where $\underline{P}_s$, $\underline{Q}_s$, $\underline{G}_s$, are the integrals of the R.H.S. of Eqs. (31), and the quantities in brackets are to be evaluated between the limits f and f_o.

In the perturbation equation for σ, Eq. (25) it may be noted that

$$\underline{R} \cdot \underline{F} = -3\,\Phi \qquad \dot{\underline{R}} \cdot \underline{F} = \frac{d\Phi}{dt}$$

If the parameters a and σ are held constant at their initial values

$$\frac{d}{dt}\left\{-\frac{3a_o}{\mu}\,\Phi\,(t - \sigma_o)\right\} = \left\{-\frac{3a_o}{\mu}\,\dot{\underline{R}}\,(t - \sigma_o) + \frac{a_o\underline{R}}{\mu}\right\} \cdot \underline{F} \tag{49}$$

Therefore, Eq. (25) may be rewritten in the form

$$\frac{d}{dt}\left\{\sigma + \frac{3a_o}{\mu}\,\Phi\,(t - \sigma_o)\right\}$$

$$= \frac{a^2}{p^2}\left\{(1 - e^2)\,(\dot{\underline{R}} \cdot \underline{R})\,\dot{\underline{R}} - \frac{r}{a}\,\underline{P}\right\} \cdot \underline{F} \tag{50}$$

Differentiation with respect to time is transformed to differentiation with respect to true anomaly, and the R.H.S. is expressed as a Fourier polynomial. The result is

$$
\begin{aligned}
\frac{d}{dt}\left\{\sigma + \frac{3a_o}{\mu}\,\Phi\,(t - \sigma_o)\right\} = \frac{3\mu^2 K_2 a}{g^3 p}\ \Bigg\{ -\left(\frac{P_3}{p}\right)^2 \Bigg[\frac{5e^2}{16}\cos 5\,f \\
+ \frac{3e}{2}\cos 4\,f + \left(\frac{5e^2}{16} + \frac{7}{4}\right)\cos 3\,f + \frac{3e}{2}\cos 2\,f + \left(\frac{5}{4} - \frac{5e^2}{8}\right)\cos f\Bigg] \\
+ \left(\frac{Q_3}{q}\right)^2 \Bigg[\frac{5e^2}{16}\cos 5\,f + \frac{3e}{2}\cos 4\,f + \left(\frac{7}{4} - \frac{7}{16}\,e^2\right)\cos 3\,f \\
- \frac{3e}{2}\cos 2\,f + \left(\frac{e^2}{8} - \frac{7}{4}\right)\cos f\Bigg] - \frac{P_3}{p}\,\frac{Q_3}{q}\Bigg[\frac{5e^2}{8}\sin 5\,f - \frac{e}{2}\sin 4\,f \\
+ \left(\frac{e^2}{8} - \frac{7}{4}\right)\cos f\Bigg] - \frac{P_3}{p}\,\frac{Q_3}{q}\Bigg[\frac{5e^2}{8}\sin 5\,f - \frac{e}{2}\sin 4\,f + \left(\frac{7}{2} - \frac{e^2}{8}\right)\sin 3\,f \\
+ 3e\sin 2\,f - \left(\frac{3e^2}{4} + \frac{1}{2}\right)\sin f\Bigg] + \Bigg[\frac{e^2}{4}\cos 3\,f + e\cos 2\,f \\
+ \left(1 - \frac{e^2}{4}\right)\cos f\Bigg]\Bigg\}
\end{aligned}
\tag{51}
$$

Holding the parameters on the R.H.S. constant, Eq. (51) is integrated to yield

$$
\sigma = \sigma_o + \left[\sigma_s - \frac{3a_o\Phi}{\mu}\,(t - \sigma_o)\right]_{f_o}^{f} \tag{52}
$$

where σ_s is the integral of the second member of Eq. (51).

CONCLUSION

The solution (47) obtained has f appearing in arguments of sines and cosines, these terms having two essentially different periods: $2\pi/j$ (short period where j is a natural number), and $2\pi/A$ (long period where A is a small quantity and equals $3\mu^2 K_2/g^4$). The solution is well behaved for all values of f because f appears in arguments of sines and cosines and because these functions are found only in the numerator. This would not be the case if Eqs. (31) were integrated keeping all parameters constant; for then, the long periodic terms in the previous solution would be replaced by their first order approximations. This solution would grow linearly with time.

The next step in the usual procedure for deriving the second order approximation consists in substituting the first order solution for the parameters in Eqs. (31). Before this step can be carried out, however, it should be recalled that Eqs. (31) were obtained by putting $dt/df = r^2/g$. If higher order solutions are to be found, this approximation is no longer valid. Therefore, for a second order approximation, dt/df must be replaced by its first order approximation derived from Eq. (28).

Now suppose the parameters are replaced by their first order solutions, terms of order $K_2{}^3$ are neglected, and products of trigonometric functions are replaced by trigonometric functions of sums. Under the following conditions, the resulting equations may be integrated to give a well behaved second order solution:

(a) No constant terms are present

(b) Whenever $\cos \alpha f$ or $\sin \alpha f$ occurs (α a small quantity), α must also appear as a factor in the numerator.

If these conditions are not fulfilled, and the equations are integrated, f may occur outside trigonometric functions, or small divisors may be present. A possible solution to these difficulties is obtained as follows:

(a) Denote the short periodic terms of the first order solution of $\underline{P}$, $\underline{Q}$, $\underline{G}$ by $\underline{P}_s(\underline{P}_o, \underline{Q}_o, f)$, $\underline{Q}_s(\underline{P}_o, \underline{Q}_o, f)$, $\underline{G}_s(\underline{P}_o, \underline{Q}_o, f)$ and assume a solution of the form $\underline{P} = \underline{P}_\ell + \underline{P}_s(\underline{P}_\ell, \underline{Q}_\ell, f)$, $\underline{Q} = \underline{Q}_\ell + \underline{Q}_s(\underline{P}_\ell, \underline{Q}_\ell, f)$, $\underline{G} = \underline{G}_\ell + \underline{G}_s(\underline{P}_\ell, \underline{Q}_\ell, f)$ $\underline{P}_\ell$, $\underline{Q}_\ell$, $\underline{G}_\ell$ are new variables which, to first order, are equivalent to solution (47).

(b) Substitute these expressions into both sides of Eqs. (31) as modified in accordance with the qualification regarding dt/df mentioned above. Neglect terms of order $K_2{}^3$; expand into Fourier polynomials, and neglect terms multiplied by sines and cosines. $\underline{P}_\ell$, $\underline{Q}_\ell$, $\underline{G}_\ell$ are determined from the resulting equations.

Investigations are currently being pursued for the purpose of finding the second order solution by this method.

APPENDIX

EXAMPLE OF RAPIDLY VARYING PARAMETERS

Whenever perturbation equations for a set of parameters are solved employing an approximate integration method, it is always desirable that the parameters be slowly varying. It is likely that, for the polar oblateness problem, no set of parameters exist in which all elements possess this characteristic. An example is presented to demonstrate the existence of rapidly varying parameters for the polar oblateness problem. Consider the equation

$$\ddot{z} + \frac{\mu z}{r^3} = - \frac{3\mu K_2}{r^5} \left[\left(1 - 5 \frac{z^2}{r^2}\right) z + 2 z \right]$$

which is obtained by taking the scalar product of Eq. (30) with $\underline{k}$. Given the initial conditions $z(t_o) = \dot{z}(t_o) = 0$, it follows that all derivatives of z evaluated at $t - t_o$ are zero. Therefore z is identically zero.

In the following example it is to be assumed that this is the case. Then $\dot{\underline{G}} = \underline{R} \times \underline{F} = 0$ or $\underline{G} = G_3 \underline{k}$ where G_3 is a constant. Now introduce a polar coordinate system, (r, θ) in the x-y plane. From Eq. (30) two scalar equations result:

$$\ddot{r} - r(\dot{\theta})^2 = - \frac{\mu}{r^2} - \frac{3\mu K_2}{r^4}$$

$$\frac{1}{r} \frac{d}{dt} (r^2 \dot{\theta}) = 0$$

A particular solution of these equations is given by

$$r = r_0, \quad \theta = \theta_0 \pm t\sqrt{\frac{\mu}{r_0^3} + \frac{3\mu K_2}{r_0^5}}$$

where r_0, θ_0 are constant. Since

$$g = |r^2 \dot{\theta}| = \sqrt{r_0 \mu + \frac{3\mu K_2}{r_0}}$$

and

$$e \cos f = \frac{g^2}{\mu r_0} - 1$$

it follows that

$$e \cos f = \frac{3K_2}{r_0^2}$$

Also, $\dot{r} = 0$, so that

$$\underline{R} \quad \dot{\underline{R}} = r\,\dot{r} = \frac{\mu r_0 e \sin f}{g} = 0$$

As a result it is seen that $e \sin f = 0$. Therefore, it may be concluded that $e > 0$, $f \equiv 0$. From the equation

$$\underline{R} = r\left(\cos f \frac{\underline{P}}{p} + \sin f \frac{\underline{Q}}{q}\right)$$

one obtains

$$\underline{R} = r\frac{\underline{P}}{p}$$

It is clear that the vector $\underline{P}$ is always in the direction of the vector $\underline{R}$ and is thus a rapidly varying parameter. Consequently, there is no guarantee that the method of variation of parameters and an approximate integration procedure will yield a satisfactory solution.

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REPUBLIC AVIATION CORPORATION

A FIRST ORDER SOLUTION TO THE POLAR OBLATENESS PROBLEM

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SUMMARY

This report contains the development of a first order solution for the polar oblateness problem with the potential limited to the second spherical harmonic. The development begins with the equations of motion of the two-body problem. Expressions for a set of parameters are derived. The perturbation equations of these parameters for an arbitrary disturbing force are generated, applied to the oblateness force and integrated analytically to obtain the first order solution. This solution is valid for all orbits except those which are nearly rectilinear.

I - INTRODUCTION

The purpose of this report is to present the development of an improved, approximate, closed form solution to the equations of motion of a vehicle about a spheroidal earth. The nonsphericity of the earth is assumed to be characterized by the second spherical harmonic. A feature common to some solutions which have been offered is a limitation on the applicability of the solutions in the neighborhood of an inclination of 63°, due to a singularity at this inclination[1,2]. The original motive for the investigation, the results of which are presented here, was to examine the possibility of overcoming that restriction. Since the use of the argument of perigee is the immediate occasion for the presence of the critical angle of inclination, an obvious corrective measure is the choice of a set of parameters which does not incorporate that element. However, numerous other pitfalls must be avoided. Some of these are: a) indetermination of the initial value of the time of perigee passage for nearly circular orbits[3], b) degeneracy of the solution caused by the presence of the eccentricity in the denominator of the perturbation equations for nearly circular orbits[3], and c) the introduction of secular terms in elements which are clearly bounded as a result of the integration of the perturbation equations. The particular set of two-body parameters selected for the present development has been chosen so as to minimize the difficulties listed above. Neither time of perigee passage nor argument of perigee is included in the set of elements, none of the perturbation equations contain the eccentricity in the denominator and the integration process is modified so that secular terms do not occur explicitly in the equations for bounded elements. However, it should be noted that the solution is not applicable to nearly rectilinear motion.

The development is self-contained. First, expressions for the two body parameters are derived from the equations of motion, then perturbation equations for these parameters are obtained for an arbitrary disturbing force, and are then particularized to the oblateness problem. Next, these equations are integrated to obtain first order corrections. Finally, some remarks are included concerning the properties of the parameters, some general results of the second order theory, and some possible applications.

In this report, the convention is adopted that capital Latin letters represent vectors (or matrices), and small Latin letters with appropriate subscripts indicate the components of these vectors.

II - DERIVATION OF A SET OF TWO-BODY PARAMETERS

The equations of motion for a vehicle of negligible mass about a spherical earth are:

$$\ddot{R} + \frac{\mu}{r^3} R = 0 \tag{1}$$

where μ is the product of the gravitational constant and the mass of the earth, R is the position vector and r is the magnitude of R. A rectangular, inertial coordinate system is used with the equatorial plane taken as the x-y plane. The general solution to these second order, differential equations generates the vectors R and $\dot{R}$ as vector functions of six constants of integration and time. The six constants are determined by a complete set of initial conditions: vectors R_0, $\dot{R}_0$ and t_0.

From the many constants that can be derived, an independent set must be selected. For application to the oblateness problem, the following set has been chosen: U, V, g, e cos θ, e sin θ and t_0. U and V are unit orthogonal vectors which specify the plane of the motion. The parameter, g, is the magnitude of the angular momentum vector, e is the eccentricity, and θ is the angle measured from U to the perigee vector. The parameters g and e determine the shape and size of the osculating ellipse and θ gives the orientation of this ellipse in the plane. Expressions for these parameters will now be derived and their independence demonstrated.

Three of the constants are obtained immediately by crossing Eq. (1) on the left by R and integrating the result.

$$R \times \ddot{R} = 0 \tag{2}$$

$$R \times \dot{R} = G = R_0 \times \dot{R}_0 \tag{3}$$

The magnitude of this constant vector just defined is one parameter, g.

The other two parameters are contained in the unit vector G/g which may be expressed as the cross product of two orthogonal unit vectors in the plane perpendicular to G. Thus

$$\frac{G}{g} = U \times V \tag{4}$$

U is arbitrarily chosen to be in the direction of R_0; this direction is not a constant of integration and therefore not a parameter. Thus R and $\dot{R}$ can be expressed as follows:

$$R = (R \cdot U)\, U + (R \cdot V)\, V \tag{5}$$

$$\dot{R} = (\dot{R} \cdot U)\, U + (\dot{R} \cdot V)\, V \tag{6}$$

Let φ denote the angle between R and U. Then $R \cdot U = r \cos \varphi$ and $R \cdot V = r \sin \varphi$.

To obtain an expression in φ for $\dot{R} \cdot U$ and $\dot{R} \cdot V$, one proceeds as follows:

$$\ddot{R} \cdot U = -\frac{\mu\,(R \cdot U)}{r^3} = -\frac{\mu}{r^3}\left[R \cdot \left(V \times \frac{G}{g}\right) \right]$$

$$= \frac{\mu}{g} V \cdot \left[R \,\frac{(R \cdot \dot{R})}{r^3} - \frac{\dot{R}}{r} \right]$$

Integration yields

$$\dot{R} \cdot U = -\frac{\mu}{g} V \cdot \frac{R}{r} + c_1 \tag{7}$$

Using the initial conditions to evaluate the constant, one obtains, recalling that initially U is in the direction of R_0,

$$\frac{\dot{R}_0 \cdot R_0}{r_0} + \frac{\mu}{g} \sin \varphi_0 = c_1 = \frac{\dot{R}_0 \cdot R_0}{r_0}$$

since $\varphi_0 = 0$. Eq. (7) now becomes

$$\dot{R} \cdot U = -\frac{\mu}{g}\left[\frac{R \cdot V}{r} - \frac{g}{\mu} \frac{\dot{R}_0 \cdot R_0}{r_0}\right] \tag{8}$$

In the same way, starting from

$$\ddot{R} \cdot V = -\frac{\mu}{g} U \cdot \left[R \frac{(R \cdot \dot{R})}{r^3} - \frac{\dot{R}}{r}\right]$$

one gets

$$\dot{R} \cdot V - \frac{\mu}{g} \frac{(R \cdot U)}{r} = c_2 \tag{9}$$

Using

$$\dot{R} \cdot \left(\frac{G}{g} \times U\right) = \frac{G \cdot G}{gr},$$

and the initial conditions, the constant c_2 may be evaluated

$$\frac{g}{r_0} - \frac{\mu}{g} = c_2 = \frac{\mu}{g}\left(\frac{g^2}{\mu r_0} - 1\right)$$

Thus, finally, Eq. (9) becomes

$$\dot{R} \cdot V = \frac{\mu}{g}\left[\frac{R \cdot U}{r} + \left(\frac{g^2}{\mu r_0} - 1\right)\right] \tag{10}$$

It is still necessary to express r in terms of φ in the new parameters. This will also allow one to express $\dot{R}$ as a function of these parameters alone. First, one multiplies Eq. (8) by $\sin\varphi$ and Eq. (10) by $\cos\varphi$ and subtracts to obtain

$$\dot{R} \cdot (U \sin\varphi - V \cos\varphi) = -\frac{\mu}{g}\left\{\frac{R}{r} \cdot (U \cos\varphi + V \sin\varphi) - \left[\frac{g}{\mu}\frac{\dot{R}_0 \cdot R_0}{r_0} \sin\varphi - \left(\frac{g^2}{\mu r_0} - 1\right)\cos\varphi\right]\right\}$$

But

$$\dot{R} \cdot (U \sin\varphi - V \cos\varphi) = \dot{R} \cdot \left(\frac{R}{r} \times \frac{G}{g}\right) = -\frac{g}{r}$$

and

$$\frac{R}{r} \cdot (U \cos\varphi + V \sin\varphi) = 1$$

Hence

$$-\frac{g}{r} = -\frac{\mu}{g}\left\{1 - \left[\frac{g}{\mu}\frac{\dot{R}_0 \cdot R_0}{r_0} \sin\varphi - \left(\frac{g^2}{\mu r_0} - 1\right)\cos\varphi\right]\right\}$$

and

$$r = \frac{g^2}{\mu\left[1 + \cos\varphi\left(\frac{g^2}{\mu r_0} - 1\right) - \frac{g}{\mu}\frac{\dot{R}_0 \cdot R_0}{r_0} \sin\varphi\right]} \tag{11}$$

One now defines the parameters $e \sin\theta$ and $e \cos\theta$ by the relations

$$\frac{g^2}{\mu r_0} - 1 = e \cos\theta$$

and

$$-\frac{g}{\mu}\frac{\dot{R}_0 \cdot R_0}{r_0} = e \sin\theta$$

where it is evident that e is the eccentricity and θ is the angle measured from U to perigee. Eq. (11) thus becomes

$$r = \frac{g^2}{\mu\left(1 + e\cos(\varphi - \theta)\right)} = \frac{g^2}{\mu\,(1 + e\cos f)} \tag{11'}$$

where f is the true anomaly.

One can now rewrite Eqs. (8) and (10) as follows:

$$\dot{R} \cdot U = -\frac{\mu}{g}(\sin\varphi + e\sin\theta) \tag{8'}$$

$$\dot{R} \cdot V = \frac{\mu}{g}(\cos\varphi + e\cos\theta) \tag{10'}$$

A further expression is required to relate φ and t. In the process of deriving this relation, a sixth constant of integration will arise. To do this one proceeds by multiplying Eq. (8) by $\cos\varphi$ and Eq. (10) by $\sin\varphi$ and adding:

$$\dot{R} \cdot (U\cos\varphi + V\sin\varphi) = -\frac{\mu}{g}\left\{\frac{R}{r} \cdot (V\cos\varphi - U\sin\varphi) - \left[\frac{g}{\mu}\frac{\dot{R}_0 \cdot R_0}{r_0}\cos\varphi + \left(\frac{g^2}{\mu r_0} - 1\right)\sin\varphi\right]\right\}$$

The left hand side is $\frac{\dot{R} \cdot R}{r}$ and the first term on the right is zero. Hence,

$$\frac{\dot{R} \cdot R}{r} = \frac{\mu}{g}\left[\frac{g}{\mu}\frac{\dot{R}_0 \cdot R_0}{r_0}\cos\varphi + \left(\frac{g^2}{\mu r_0} - 1\right)\sin\varphi\right] = \frac{\mu}{g}e\sin f = \dot{r} \tag{12}$$

From Eq. (11)

$$\dot{r} = -\frac{g^2\left[-\sin\varphi\left(\frac{g^2}{\mu r_0} - 1\right) - \frac{g}{\mu}\frac{\dot{R}_0 \cdot R_0}{r_0}\cos\varphi\right]}{\mu\left[1 + \cos\varphi\left(\frac{g^2}{\mu r_0} - 1\right) - \frac{g}{\mu}\frac{\dot{R}_0 \cdot R_0}{r_0}\sin\varphi\right]^2}\dot{\varphi} = \frac{\mu}{g^2}r^2\left[\frac{\dot{R} \cdot R}{r}\right]\frac{g}{\mu}\dot{\varphi}$$

or

$$\dot{r} = \frac{R \cdot \dot{R}}{r} = \frac{1}{g} r^2 \left(\frac{\dot{R} \cdot R}{r} \right) \dot{\varphi}$$

Hence

$$\dot{\varphi} = \frac{g}{r^2} \tag{13}$$

This equation may be written as

$$r\dot{\varphi} = \frac{g}{r}$$

and using Eq. (11′), one obtains

$$\frac{g^2}{\mu \left[1 + e \cos f\right]} \dot{\varphi} = \frac{\mu}{g} (1 + e \cos f)$$

or

$$\frac{\dot{\varphi}}{1 + e \cos f} = \frac{\mu^2}{g^3} \left[1 - e^2 + e \cos f + e^2 \cos^2 f + e^2 \sin^2 f\right] \tag{14}$$

Considering only the factor on the right, the following statements can be made.

(1) $\frac{\mu^2}{g^3} (1-e^2)$ is a constant and can be shown to be equal to $\frac{n}{\sqrt{1 - e^2}}$

where n is the mean motion

(2) $\frac{1}{g} \frac{\mu^2}{g^2} e \cos f \, (1+e \cos f) = \frac{\mu}{g} \frac{e \cos f}{r} \frac{r g}{r g} = \frac{1}{g} \frac{\mu r}{g} \frac{d}{dt} e \sin f$

(3) $\frac{1}{g} \frac{\mu^2}{g^2} e^2 \sin^2 f = \frac{1}{g} \frac{\mu}{g} e \sin f \, \dot{r}$

Combining the last two expressions above, one gets

$$\frac{\mu}{g^2} \left[r \frac{d}{dt} (e \sin f) + e \sin f \, \dot{r}\right] = \frac{\mu}{g^2} \frac{d}{dt} (r e \sin f)$$

Thus Eq. (14) now becomes

$$\frac{\dot{\varphi}}{1 + e \cos f} = \frac{n}{\sqrt{1 - e^2}} + \frac{\mu}{g^2} \frac{d}{dt} (r\ e \sin f)$$

Integrating with respect to t, the left hand side becomes after some algebraic reduction,

$$\int \frac{\dot{\varphi}\, dt}{1 + e \cos f} = \frac{2}{\sqrt{1 - e^2}} \tan^{-1} \left(\frac{\sqrt{1 - e^2} \sin \varphi}{(1 + \cos \varphi)(1 + e \cos f) + e \sin f \sin \varphi} \right)$$

and the final equation* is

$$n\,(t - t_0) = 2 \tan^{-1} \left(\frac{\sqrt{1 - e^2} \sin \varphi}{(1 + \cos \varphi)(1 + e \cos f) + e \sin f \sin \varphi} \right)$$

$$- \frac{\mu}{g^2} \sqrt{1 - e^2}\ (r\ e \sin f + r_o\ e \sin \theta) \qquad (15)^*$$

This is also the defining equation for t_0, the sixth constant of integration.

In Eqs. (5), (6) and (15) the constants U, V, g, e sin θ, e cos θ and t_0 occur. To summarize, these constants are defined by the following equations:

$$U = \frac{R}{r} \cos \varphi - \sin \varphi \left[\frac{G}{g} \times \frac{R}{r} \right] \qquad (16)$$

$$V = \frac{R}{r} \sin \varphi + \cos \varphi \left[\frac{G}{g} \times \frac{R}{r} \right] \qquad (17)$$

$$g = |\,G\,| \qquad (18)$$

$$e \cos \theta = \frac{g}{\mu} (\dot{R} \cdot V) - \cos \varphi \qquad (19)$$

---

* This equation holds only for e < 1. Only slight modifications are required required for e ≥ 1.

$$e \sin \theta = \frac{g}{\mu} (\sin \varphi) - \dot{R} \cdot U \tag{20}$$

$$t_0 = t - \frac{2}{n} \tan^{-1} \left(\frac{\sqrt{1 - e^2} \sin \varphi}{(1 + \cos \varphi)(1 + e \cos f) + e \sin f \sin \varphi} \right)$$

$$+ \frac{\mu}{ng^2} \sqrt{1 - e^2} \, (r \, e \sin f + r_0 \, e \sin \theta) \tag{21}$$

It should be noted that U and V account for only two independent parameters since they are orthogonal unit vectors and the direction of U was chosen arbitrarily. It remains to be shown that the parameters just defined are independent of each other. This will be proved by showing the equivalence between the set above and the set R_0, $\dot{R}_0$ whose elements are known to be independent of each other. Further discussion of these parameters appears in Section VI.

From the derivation that has preceded, one easily obtains R_0 and $\dot{R}_0$ in terms of the parameters on the one hand, namely,

$$r_0 = \frac{g^2}{\mu (1 + e \cos \theta)}$$

$$R_0 = \frac{g^2}{\mu (1 + e \cos \theta)} U$$

$$\dot{R}_0 = \left(\frac{\mu}{g} - e \sin \theta \right) U + (1 + e \cos \theta) V$$

and on the other hand, the following parameters in terms of R_0 and $\dot{R}_0$:

$$U = \frac{R_0}{r_0}$$

$$g = \sqrt{(R_0 \times \dot{R}_0) \cdot (R_0 \times \dot{R}_0)}$$

$$V = \frac{G}{g} \times \frac{R_0}{r_0} \qquad (G = R_0 \times \dot{R}_0)$$

$$e \cos \theta = \frac{g^2}{\mu r_0} - 1$$

$$e \sin \theta = -\frac{g}{\mu} \frac{R_0 \cdot \dot{R}_0}{r_0}$$

$$t_0 = t_0$$

III - PERTURBATION EQUATIONS

Before proceeding to the development of the perturbation equations, it should be observed that, of the quantities listed at the end of the preceding section, only six have been obtained as constants of integration. These are t_0, $e \sin \theta$, $e \cos \theta$, g, and two contained in U and V which determine the plane of U and V. The third constant contained in U and V which specifies the direction of U in the plane is arbitrary. This last arbitrary constant does not vary under the action of the disturbing force. As a result, U is not subject to rotation about the angular momentum vector.* Since φ is measured from U, this restriction implies that $\dot{\varphi}$ does not include the time rate of change of U in the osculating plane. As a consequence, the time rate of change of φ must have the same functional form that it has in a purely Keplerian motion, i.e., $\dot{\varphi} = g/r^2$. Keeping in mind the result just noted, the method for obtaining the perturbation equations for the set of parameters is as follows:

- Each of the Eqs. (16-21) is differentiated with respect to time (considering the parameters too as functions of time)
- Wherever $\ddot{R}$ occurs, it is replaced by

$$-\frac{\mu R}{r^3} + F \tag{22}$$

  where F is the disturbing force
- The resulting equations are simplified by making use of the relations obtained in the preceding section.

* Compare Ref. (4).

1. <u>Equation for $\dot{g}$</u>

Noting that

$$g^2 = G \cdot G \qquad \dot{g} = \frac{G}{g} \cdot \dot{G} \qquad G = R \times \dot{R}$$

one obtains

$$\dot{G} = R \times \ddot{R} = R \times \left(- \mu \frac{R}{r^3} + F \right) = R \times F \tag{23}$$

and hence

$$\dot{g} = \left(\frac{G}{g} \times R \right) \cdot F \tag{24}$$

2. <u>Equation for $\dot{U}$</u>

Differentiating Eq. (16) with respect to time

$$\dot{U} = \left(- \sin \varphi \frac{R}{r} - \cos \varphi \frac{G}{g} \times \frac{R}{r} \right) \dot{\varphi} - \frac{R \cdot \dot{R}}{r^2} U$$

$$+ \cos \varphi \frac{\dot{R}}{r} - \sin \varphi \left(\frac{G}{g} \times \frac{\dot{R}}{r} \right) - \sin \varphi \left[\left(\frac{\dot{G}}{g} \right) \times \frac{R}{r} \right]$$

Then, using

$$\dot{R} = \frac{\mu}{g} \left[- (\sin \varphi + e \sin \theta) U + (\cos \varphi + e \cos \theta) V \right]$$

and

$$\frac{G}{g} \times \frac{\dot{R}}{r} = - \frac{\mu}{gr} \left[(\sin \varphi + e \sin \theta) V + (\cos \varphi + e \cos \theta) U \right]$$

it follows that

$$\cos\varphi \frac{\dot{R}}{r} - \sin\varphi\left(\frac{G}{g} \times \frac{\dot{R}}{r}\right) = \frac{\mu}{gr}\left[e \sin f\, U + (1 + e\cos f)\, V\right]$$

But

$$\frac{\mu}{gr} e \sin f = \frac{R \cdot \dot{R}}{r^2} \qquad \text{(Eq. (12))}$$

and

$$\frac{\mu}{gr}(1 + e\cos f) = \frac{g}{r^2} \qquad \text{(Eq. (11'))}$$

or

$$\frac{\mu}{gr}(1 + e\cos f) = \dot{\varphi}$$

The coefficient of $\dot{\varphi}$ in the $\dot{U}$ equation above is then simply $- V$. Replacing these terms in the $\dot{U}$ equation

$$\dot{U} = - V\dot{\varphi} - \frac{R \cdot \dot{R}}{r^2} U + \frac{R \cdot \dot{R}}{r^2} U + V\dot{\varphi} - \sin\varphi\left[\frac{d}{dt}\left(\frac{G}{g}\right) \times \frac{R}{r}\right]$$

but

$$\frac{d}{dt}\left(\frac{G}{g}\right) = - \frac{G}{g} \times \frac{R}{g}\left(F \cdot \frac{G}{g}\right)$$

and hence

$$\dot{U} = \sin\varphi\left[\left(\frac{G}{g} \times \frac{R}{g}\right) \times \frac{R}{r}\right]\left(F \cdot \frac{G}{g}\right)$$

or

$$\dot{U} = - \sin\varphi \frac{r}{g}\left(\frac{G}{g} \cdot F\right)\frac{G}{g} \qquad (25)$$

3. <u>Equation for $\dot{V}$</u>

In an entirely analogous fashion one obtains $\dot{V}$.

$$\dot{V} = \cos\varphi \frac{r}{g}\left(\frac{G}{g} \cdot F\right)\frac{G}{g} \tag{26}$$

4. Equation for $\dot{(e \cos \theta)}$

$$\dot{(e \cos \theta)} = \frac{g}{\mu}\ddot{R} \cdot V + \sin\varphi\,\dot{\varphi} + \frac{\dot{g}}{\mu}\dot{R} \cdot V + \frac{g}{\mu}\dot{R} \cdot \dot{V}$$

From Eq. (26) it follows that

$$\dot{R} \cdot \dot{V} = 0$$

and, rearranging terms,

$$\dot{(e \cos \theta)} = \left(-\frac{g}{r^2}\sin\varphi + \dot{\varphi}\sin\varphi\right) + \frac{g}{\mu}F \cdot V + \frac{\dot{g}}{\mu}\dot{R} \cdot V$$

The first term on the right is identically zero, so that, finally

$$\dot{(e \cos \theta)} = \frac{g}{\mu}F \cdot V + \frac{\dot{g}}{\mu}\dot{R} \cdot V \tag{27}$$

5. Equation for $\dot{(e \sin \theta)}$

In analogous fashion

$$\dot{(e \sin \theta)} = -\left[\frac{\dot{g}}{\mu}\dot{R} \cdot U + \frac{g}{\mu}F \cdot U\right] \tag{28}$$

6. Equation for $\dot{t}_0$

In order to simplify the derivation of $\dot{t}_0$, Kepler time derivatives will be assumed to have been canceled. In addition, in order to simplify the writing,

$$e \cos f = p \qquad e \cos \theta = p_o$$
$$e \sin f = q \qquad -e \sin \theta = q_o \tag{29}$$

' Starting from Eq. (15)

$$n\,(t-t_0) = 2\tan^{-1}\left[\frac{\sqrt{1-e^2}\sin\frac{\varphi}{2}}{\cos\frac{\varphi}{2}(1+p)+q\sin\frac{\varphi}{2}}\right] - \sqrt{1-e^2}\left(\frac{q}{1+p}-\frac{q_o}{1+p_o}\right)$$

one obtains, on differentiating with respect to time,

$$\dot{n}\,(t-t_0) - n\,\dot{t}_0 = \frac{2}{1+\frac{(1-e^2)\sin^2\frac{\varphi}{2}}{\left[\cos\frac{\varphi}{2}(1+p)+q\sin\frac{\varphi}{2}\right]^2}}\frac{d}{dt}\left(\frac{\sin\frac{\varphi}{2}\sqrt{1-e^2}}{\cos\frac{\varphi}{2}(1+p)+q\sin\frac{\varphi}{2}}\right)$$

$$-\left(\frac{q}{1+p}-\frac{q_o}{1+p_o}\right)\frac{d}{dt}\left(\sqrt{1-e^2}\right) - \sqrt{1-e^2}\,\frac{d}{dt}\left(\frac{q}{1+p}-\frac{q_o}{1+p_o}\right) \tag{30}$$

a. The first term on the right of Eq. (30) becomes

$$2\,\frac{\left[\sin\frac{\varphi}{2}\left(\cos\frac{\varphi}{2}(1+p)+q\sin\frac{\varphi}{2}\right)\frac{d}{dt}\left(\sqrt{1-e^2}\right) - \sqrt{1-e^2}\sin\frac{\varphi}{2}\left(\dot{p}\cos\frac{\varphi}{2}+\dot{q}\sin\frac{\varphi}{2}\right)\right]}{\left[\cos\frac{\varphi}{2}(1+p)+q\sin\frac{\varphi}{2}\right]^2+(1-e^2)\sin^2\frac{\varphi}{2}}$$

The denominator, by making use of the identity $p^2+q^2=e^2$, is

$$(1+p)\,(1+p_o)$$

b. The last term in Eq. (30) is

$$-\sqrt{1-e^2}\left(\frac{\dot{q}\,(1+p)-q\dot{p}}{(1+p)^2}-\frac{\dot{q}_o\,(1+p_o)-\dot{p}_o\,q_o}{(1+p_o)^2}\right)$$

c. The right hand side of Eq. (30) thus becomes

$$-\sqrt{1-e^2}\left\{\frac{2\sin\frac{\varphi}{2}\left(\dot{p}\cos\frac{\varphi}{2}+\dot{q}\sin\frac{\varphi}{2}\right)}{(1+p)\,(1+p_o)}+\frac{\dot{q}\,(1+p)-q\,\dot{p}}{(1+p)^2}-\frac{\dot{q}_o\,(1+p_o)-\dot{p}_o\,q_o}{(1+p_o)^2}\right\}$$

$$+\frac{\frac{d}{dt}\left(\sqrt{1-e^2}\right)}{(1+p)\,(1+p_o)}\left\{2\sin\frac{\varphi}{2}\left[(1+p)\cos\frac{\varphi}{2}+q\sin\frac{\varphi}{2}\right]-q\,(1+p_o)+q_o\,(1+p)\right\}$$

d. $\frac{d}{dt}\sqrt{1-e^2} = -\frac{(p\dot{p} + q\dot{q})}{\sqrt{1-e^2}}$

The coefficient of this derivative becomes, after some algebraic manipulations, $(1 - e^2)\sin\varphi$ and one gets for the whole equation,

$$\dot{n}(t-t_0) - n\dot{t}_0 = -\sqrt{1-e^2}\left\{\frac{(p\dot{p}+q\dot{q})\sin\varphi + \dot{p}\sin\varphi + (1-\cos\varphi)\dot{q}}{(1+p)\ (1+p_o)}\right.$$

$$\left. + \frac{(1+p)\dot{q} - q\dot{p}}{(1+p)^2} - \frac{\dot{q}_o(1+p_o) - \dot{p}_o q_o}{(1+p_o)^2}\right\}$$

e. Using the fact that

$$p_o = p\cos\varphi + q\sin\varphi$$

$$q_o = -p\sin\varphi + q\cos\varphi$$

$$\dot{p}_o = \dot{p}\cos\varphi + \dot{q}\sin\varphi$$

$$\dot{q}_o = -\dot{p}\sin\varphi + \dot{q}\cos\varphi$$

$$-\frac{\dot{q}_o(1+p_o) - \dot{p}_o q_o}{(1+p_o)^2} = \frac{(\dot{p}\sin\varphi - \dot{q}\cos\varphi)(1+p_o) + (\dot{p}\cos\varphi + \dot{q}\sin\varphi)q_o}{(1+p_o)^2}$$

and rearranging, one gets

$$\dot{n}(t-t_0) - n\dot{t}_0 = \frac{-\sqrt{1-e^2}}{(1+p)^2\ (1+p_o{}^2)}\left\{\dot{p}\Big[(p\sin\varphi + \sin\varphi)(1+p)(1+p_o)\right.$$

$$- q(1+p_o)^2 + (\sin\varphi(1+p_o) + q_o\cos\varphi)(1+p)^2\Big]$$

$$+ \dot{q}\Big[(q\sin\varphi + 1 - \cos\varphi)(1+p)(1+p_o) + (1+p)(1+p_o)^2$$

$$\left. + (q_o\sin\varphi - \cos\varphi(1+p_o))(1+p)^2\Big]\right\} \qquad (31)$$

f. Noting that

$$p = \frac{g^2}{\mu r} - 1 \qquad q = \frac{g}{\mu} \frac{R \cdot \dot{R}}{r}$$

$$\dot{p} = \frac{2\dot{g}}{g}(1+p) \qquad \dot{q} = \frac{\dot{g}}{g} q + \frac{(1+p)}{g} R \cdot F \tag{32}$$

it is now possible to express Eq. (31) in a more convenient form. First, the right-hand (R. H.) side of Eq. (31) is rearranged as follows:

$$\text{R. H.} = \frac{-\sqrt{1-e^2}}{(1+p)^2 (1+p_o^{\,2})} \Big\{ \frac{\dot{g}}{g}(1+p)\Big[2 \sin\varphi (1+p)^2 (1+p_o) - q (1+p_o)^2$$

$$+ 2\big(\sin\varphi (1+p_o) + q_o \cos\varphi\big)(1+p)^2 + q (1+p_o)(q \sin\varphi + 1 - \cos\varphi)$$

$$+ \big(q q_o \sin\varphi - q \cos\varphi (1+p_o)\big)(1+p)\Big]$$

$$+ (1+p)^2 \frac{R \cdot F}{g}\Big[(q \sin\varphi + 1 - \cos\varphi)(1+p_o) + (1+p_o)^2$$

$$+ (q_o \sin\varphi - \cos\varphi (1+p_o))(1+p)\Big]\Big\}$$

1) The coefficient of $\dot{g}$ is rewritten as follows:

$$2 \sin\varphi (1+p)^2 (1+p_o) + 2(\sin\varphi (1+p_o) + q_o \cos\varphi)(1+p)^2 + \big(q q_o \sin\varphi$$

$$- q \cos\varphi (1+p_o)\big)(1+p)$$

$$-(1+p_o)(q + q p_o - q^2 \sin\varphi - q + q \cos\varphi)$$

Replacing p_o by $p \cos\varphi + q \sin\varphi$, one can factor out $(1+p)$ from the whole expression and rearranging again,

$$(1+p)\left\{2(1+p_o)\sin\varphi\,(2+p) + (1+p_o)(2p\sin\varphi - 2q\cos\varphi) + 2(1+p)q_o\cos\varphi + q\,q_o\sin\varphi\right\}$$

But

$$-p\sin\varphi + q\cos\varphi = q_o \tag{33}$$

and

$$q\sin\varphi + p\cos\varphi = p_o \tag{34}$$

thus by Eq. (34)

$$(1+p)\left\{2(1+p_o)\sin\varphi\,(2+p) - 2(1+p_o)q_o + 2(1+p)q_o\cos\varphi + q\,q_o\sin\varphi\right\}$$

$$= (1+p)\left[2\left\{\sin\varphi\,(1+p_o)(2+p) - q_o(1-\cos\varphi)\right\} - q\,q_o\sin\varphi\right]$$

2) The coefficient of $R \cdot F$ is rewritten as follows:

$$2(1+p_o)(q\sin\varphi + 1 - \cos\varphi) + (1+p_o)\left[1+p_o - q\sin\varphi - 1 + \cos\varphi - \cos\varphi - p\cos\varphi\right] + (1+p)q_o\sin\varphi$$

By using Eq. (34) this expression reduces to

$$2(1+p_o)(q\sin\varphi + 1 - \cos\varphi) + q_o\sin\varphi\,(1+p)$$

Putting the last two results together, one gets

$$R.H. = \frac{-\sqrt{1-e^2}}{(1+p_o)^2 g}\left\{\dot g\left[2\left\{\sin\varphi(1+p_o)(2+p) - q_o(1-\cos\varphi)\right\} - q\,q_o\sin\varphi\right] + R \cdot F\left[2(1+p_o)(q\sin\varphi + 1 - \cos\varphi) + (1+p)q_o\sin\varphi\right]\right\}$$

g. $\dot n\,(t-t_0) - n\,(\dot t_0)$

Noting that

$$n = \sqrt{\frac{\mu}{a^3}} \qquad \frac{1}{a} = \frac{2}{r} - \frac{\dot{R} \cdot \dot{R}}{\mu}$$

$$\dot{n} = -\frac{3}{2} n \frac{\dot{a}}{a} \qquad \dot{a} = \frac{2a^2}{\mu} \dot{R} \cdot F$$

thus

$$\dot{n} = -3 \frac{n\,a}{\mu} \dot{R} \cdot F$$

and

$$\dot{n}\,(t-t_0) - n\,(\dot{t}_0) = -n\left[\frac{3a}{\mu} \dot{R} \cdot F\,(t-t_0) + \dot{t}_0\right]$$

If both sides of Eq. (31) are then divided by n, the coefficient of R. H. becomes

$$-\frac{\sqrt{1-e^2}}{g\,n\,(1+p_o)^2}$$

but

$$\frac{\sqrt{1-e^2}}{g} = \frac{1}{\sqrt{\mu a}}$$

and the coefficient becomes

$$-\frac{1}{(1+p_o)^2} \sqrt{\frac{a^3}{\mu^2 a}} = -\frac{1}{(1+p_o)^2} \frac{a}{u}$$

Thus, Eq. (31) becomes finally

$$\dot{t}_0 + \frac{3a}{\mu} \dot{R} \cdot F\,(t-t_0) = \frac{a}{\mu(1+p_o)^2} \Big\{ \dot{g} \Big[2\{\sin\varphi\,(1+p_o)\,(2+p) - q_o\,(1-\cos\varphi)\} - q\,q_o \sin\varphi \Big]$$

$$+ (R \cdot F) \Big[2\,(1+p_o)\,(q \sin\varphi + 1 - \cos\varphi) + (1+p)\,q_o \sin\varphi \Big] \Big\} \quad (35)$$

IV - APPLICATION OF THE PERTURBATION EQUATIONS TO THE POLAR OBLATENESS PROBLEM

For this problem, $F = \nabla\Phi$ where

$$\Phi = \frac{\mu k_2}{r^3}\left(1 - 3\frac{z^2}{r^2}\right) \tag{36}$$

and where k_2 is the coefficient of the second spherical harmonic due to the oblateness of the earth. Then

$$F = -\frac{3\mu k_2}{r^5}\left\{\left(1 - 5\frac{z^2}{r^2}\right)R + 2\,z\,K\right\} \tag{37}$$

By Euler's theorem

$$R \cdot F = -3\,\Phi = \frac{-3\,\mu\,k_2}{r^3}\left(1 - \frac{3z^2}{r^2}\right) \tag{38}$$

Since all the expressions on the right side of the perturbation equations are expressed more simply in terms of φ than in terms of t, derivatives with respect to t will be replaced by derivatives with respect to φ. For this purpose relation (14) is used, i.e.,

$$\dot{\varphi} = \frac{g}{r^2}$$

from which one obtains

$$\frac{d}{d\varphi} = \frac{dt}{d\varphi}\,\frac{d}{dt} = \left(\frac{r^2}{g}\right)\frac{d}{dt} \tag{39}$$

The right-hand sides are expanded in terms of trigonometric polynomials in multiples of φ with functions of the parameters as coefficients.

For purposes of integration the perturbation equations are all written as the sum of two parts, the first of which, indicated by a subscript H will be integrated "exactly" while the second part, indicated by a subscript S, contains short period terms only.

The perturbation equations are as follow:

1) $(e \cos \theta)' = (e \cos \theta)_H' + (e \cos \theta)_S'$

where

$$(e \cos \theta)_H' = \frac{3\mu^2 k_2}{g^4} \left[\frac{3}{2}(u_3^2 + v_3^2) - 1\right] (1 + 2 e \cos f) \sin \varphi \tag{40a}$$

$$\begin{aligned}(e \cos \theta)_S' = -\frac{3\mu^2 k_2}{g^4} \Big\{ & -\frac{1}{4}(u_3^2 - v_3^2)(7 \sin 3\varphi + \sin \varphi) + \frac{1}{2} u_3 v_3 (7 \cos 3\varphi + \cos \varphi) \\ & + e \cos \theta \left[-\frac{1}{2}(u_3^2 - v_3^2)(3 \sin 4\varphi + 5 \sin 2\varphi) + u_3 v_3 (3 \cos 4\varphi + 5 \cos 2\varphi)\right] \\ & + e \sin \theta \left[\frac{3}{2}(u_3^2 - v_3^2)(\cos 4\varphi - \cos 2\varphi) + 3 u_3 v_3 (\sin 4\varphi - \sin 2\varphi)\right] \\ & + (e \sin \theta)(e \cos \theta)\Big[-\frac{1}{2}(\cos 3\varphi - \cos \varphi) + \left(\frac{5}{8}\cos 5\varphi + \frac{9}{8}\cos 3\varphi - \frac{7}{4}\cos \varphi\right) u_3^2 \\ & + v_3^2 \left(-\frac{5}{8}\cos 5\varphi + \frac{3}{8}\cos 3\varphi + \frac{1}{4}\cos \varphi\right) + u_3 v_3 \left(\frac{5}{4}\sin 5\varphi + \frac{3}{4}\sin 3\varphi - \frac{5}{2}\sin \varphi\right)\Big] \\ & + (e \cos \theta)^2 \Big[\frac{1}{4}(\sin 3\varphi + \sin \varphi) - \frac{1}{16}(5 \sin 5\varphi + 23 \sin 3\varphi + 18 \sin \varphi) u_3^2 \\ & + \frac{1}{16}(5 \sin 5\varphi + 11 \sin 3\varphi + 6 \sin \varphi) v_3^2 + \frac{u_3 v_3}{8}(5 \cos 5\varphi + 17 \cos 3\varphi + 10 \cos \varphi)\Big] \\ & + (e \sin \theta)^2 \Big[-\frac{1}{4}(\sin 3\varphi - 3 \sin \varphi) + \frac{u_3^2}{16}(5 \sin 5\varphi - 5 \sin 3\varphi - 10 \sin \varphi) \\ & + \frac{v_3^2}{16}(-5 \sin 5\varphi + 17 \sin 3\varphi - 26 \sin \varphi) + \frac{u_3 v_3}{8}(-5 \cos 5\varphi + 11 \cos 3\varphi \\ & - 6 \cos \varphi)\Big]\Big\}\end{aligned} \tag{40b}$$

2) $(e \sin \theta)' = (e \sin \theta)_H' + (e \sin \theta)_S'$

where

$$(e\sin\theta)_H{}' = -\frac{3\mu^2 k_2}{g^4}\left[\frac{3}{2}(u_3{}^2+v_3{}^2)-1\right](1+2\ e\cos f)\cos\varphi \tag{41a}$$

$$(e\sin\theta)_S{}' = \frac{3\mu^2 k_2}{g^4}\Big\{-\frac{1}{4}(u_3{}^2-v_3{}^2)(7\cos 3\varphi-\cos\varphi)-\frac{1}{2}u_3v_3\ (7\sin 3\varphi-\sin\varphi)$$

$$+\ e\cos\theta\left[-\frac{3}{2}(u_3{}^2-v_3{}^2)(\cos 4\varphi+\cos 2\varphi)-3u_3v_3\ (\sin 4\varphi+\sin 2\varphi)\right.$$

$$+\ e\sin\theta\left[-\frac{1}{2}(u_3{}^2-v_3{}^2)(3\sin 4\varphi-5\sin 2\varphi)+u_3v_3\ (3\cos 4\varphi-5\cos 2\varphi)\right.$$

$$+\ (e\sin\theta)(e\cos\theta)\left[\frac{1}{2}(\sin 3\varphi+\sin\varphi)+\frac{u_3{}^2}{8}(-5\sin 5\varphi-3\sin 3\varphi+2\sin\varphi)\right. \tag{41b}$$

$$\left.+\frac{v_3{}^2}{8}(5\sin 5\varphi-9\sin 3\varphi-14\sin\varphi)+\frac{u_3v_3}{8}(5\cos 5\varphi-3\cos 3\varphi-10\cos\varphi)\right]$$

$$+\ (e\cos\theta)^2\left[\frac{1}{4}(\cos 3\varphi+3\cos\varphi)-\frac{u_3{}^2}{16}(5\cos 5\varphi+17\cos 3\varphi-26\cos\varphi)\right.$$

$$\left.+\frac{v_3{}^2}{16}(5\cos 5\varphi+5\cos 3\varphi-10\cos\varphi)-\frac{u_3v_3}{8}(5\sin 5\varphi+11\sin 3\varphi+6\sin\varphi)\right]$$

$$+\ (e\sin\theta)^2\left[-\frac{1}{4}(\cos 3\varphi-\cos\varphi)+\frac{u_3{}^2}{16}(5\cos 5\varphi-11\cos 3\varphi+6\cos\varphi)\right.$$

$$\left.\left.+\frac{v_3{}^2}{16}(-5\cos 5\varphi+23\cos 3\varphi-18\cos\varphi)+\frac{u_3v_3}{8}(5\sin 5\varphi-17\sin 3\varphi+10\sin\varphi\right]\right\}$$

Next, the perturbation equations for the components of U and V are given.

3) $(u_1)' = (u_1)_H{}' + (u_1)_S{}'$

$$(u_1)_H{}' = \frac{6\mu^2 k_2}{g^4}\sin\varphi\frac{g_3}{g}\left\{(-v_2+u_1\frac{g_3}{g})\cos\varphi+(u_2+v_1\frac{g_3}{g})\sin\varphi\right\} \tag{42a}$$

$$(u_1)_S{}' = \frac{6\mu^2 k_2}{g^4} (e \cos f) \frac{g_3}{g} \sin\varphi \left\{ \left(-v_2 + u_{10} \frac{g_3}{g} \right) \cos\varphi + \left(u_2 + v_1 \frac{g_3}{g} \right) \sin\varphi \right\} \tag{42b}$$

4) $(u_2)' = (u_2)_H{}' + (u_2)_S{}'$

$$(u_2)_H{}' = \frac{6\mu^2 k_2}{g^4} \sin\varphi \frac{g_3}{g} \left\{ \left(v_1 + u_2 \frac{g_3}{g} \right) \cos\varphi + \left(-u_1 + v_2 \frac{g_3}{g} \right) \sin\varphi \right\} \tag{43a}$$

$$(u_2)_S{}' = \frac{6\mu^2 k_2}{g^4} (e \cos f) \frac{g_3}{g} \sin\varphi \left\{ \left(v_1 + u_2 \frac{g_3}{g} \right) \cos\varphi + \left(-u_1 + v_2 \frac{g_3}{g} \right) \sin\varphi \right\} \tag{43b}$$

5) $(u_3)' = (u_3)_H{}' + (u_3)_S{}'$

$$(u_3)_H{}' = \frac{6\mu^2 k_2}{g^4} \left(\frac{g_3}{g} \right)^2 \sin\varphi \, (u_3 \cos\varphi + v_3 \sin\varphi) \tag{44a}$$

$$(u_3)_S{}' = \frac{6\mu^2 k_2}{g^4} \left(\frac{g_3}{g} \right)^2 (e \cos f) \sin\varphi \, (u_3 \cos\varphi + v_3 \sin\varphi) \tag{44b}$$

6) $(v_1)' = (v_1)_H{}' + (v_1)_S{}'$

$$(v_1)_H{}' = -\frac{6\mu^2 k_2}{g^4} \cos\varphi \frac{g_3}{g} \left\{ \left(-v_2 + u_1 \frac{g_3}{g} \right) \cos\varphi + \left(u_2 + v_1 \frac{g_3}{g} \right) \sin\varphi \right\} \tag{45a}$$

$$(v_1)_S{}' = -\frac{6\mu^2 k_2}{g^4} (e \cos f) \frac{g_3}{g} \cos\varphi \left\{ \left(-v_2 + u_1 \frac{g_3}{g} \right) \cos\varphi + \left(u_2 + v_1 \frac{g_3}{g} \right) \sin\varphi \right\} \tag{45b}$$

7) $(v_2)' = (v_2)_H{}' + (v_2)_S{}'$

$$(v_2)_H{}' = -\frac{6\mu^2 k_2}{g^4}\cos\varphi\,\frac{g_3}{g}\left\{\left(v_1 + u_2\frac{g_3}{g}\right)\cos\varphi + \left(-u_1 + v_2\frac{g_3}{g}\right)\sin\varphi\right\} \qquad (46a)$$

$$(v_2)_S{}' = -\frac{6\mu^2 k_2}{g^4}\,e\cos f\,\frac{g_3}{g}\cos\varphi\left\{\left(v_1 + u_2\frac{g_3}{g}\right)\cos\varphi + \left(-u_1 + v_2\frac{g_3}{g}\right)\sin\varphi\right\} \qquad (46b)$$

8) $(v_3)' = (v_3)_H{}' + (v_3)_S{}'$

$$(v_3)_H{}' = -\frac{6\mu^2 k_2}{g^4}\left(\frac{g_3}{g}\right)^2\cos\varphi\,(u_3\cos\varphi + v_3\sin\varphi) \qquad (47a)$$

$$(v_3)_S{}' = -\frac{6\mu^2 k_2}{g^4}(e\cos f)\left(\frac{g_3}{g}\right)^2\cos\varphi\,(u_3\cos\varphi + v_3\sin\varphi) \qquad (47b)$$

9) $g^3 g' = g_H{}' + g_S{}'$

$$g_H{}' = -6\mu^2 k_2\,(v_3\cos\varphi - u_3\sin\varphi)(u_3\cos\varphi + v_3\sin\varphi) \qquad (48a)$$

$$g_S{}' = -6\mu^2 k_2\,e\cos f\,(v_3\cos\varphi - u_3\sin\varphi)(u_3\cos\varphi + v_3\sin\varphi) \qquad (48b)$$

10) $\dot{t}_0$

$$\dot{t}_0 + \frac{3a}{\mu}\dot{\Phi}(t-t_0) + \frac{3a}{\mu}\Phi = -\frac{3ak_2\mu}{(1+e\cos\theta)^2 g^3}\left\{\left[\frac{3}{2}(u_3{}^2 + v_3{}^2) - 1\right](1-e^2)\right.$$

$$+\cos\varphi\left[u_3 v_3\left(-2e\sin\theta + \frac{3}{4}e\sin\theta\, e\cos\theta + \frac{7}{8}e\sin\theta\,(e\cos\theta)^2 + \frac{3}{4}(e\sin\theta)^3\right)\right.$$

$$-(u_3{}^2 - v_3{}^2)\left(\frac{1}{2} + \frac{5}{4}e\cos\theta + \frac{7}{8}(e\cos\theta)^2 + \frac{1}{8}(e\cos\theta)^3 + \frac{9}{8}(e\sin\theta)^2\right)$$

(49)

(cont'd on next page)

$$+\left(\frac{3}{2}(u_3+v_3)-1\right)\left(-2-e\cos\theta+\frac{1}{2}(e\cos\theta)^2-\frac{1}{2}(e\cos\theta)^3\right.$$

$$\left.-\frac{1}{2}(e\sin\theta)^2-e\cos\theta\,(e\sin\theta)^2\right)\Big]$$

$$+\cos 2\varphi\Big[u_3v_3\left(\frac{3}{2}e\sin\theta+\frac{5}{2}e\cos\theta\, e\sin\theta\right)+(u_3^{\,2}-v_3^{\,2})\left(-\frac{3}{2}-\frac{7}{4}e\cos\theta-\frac{1}{4}(e\cos\theta)^2\right.$$

$$\left.+\frac{5}{8}(e\sin\theta)^2\right)-\left(\frac{3}{2}(u_3^{\,2}+v_3^{\,2})-1\right)\left(e\cos\theta+(e\cos\theta)^2\right.$$

$$\left.+\frac{1}{2}(e\sin\theta)^2\right)\Big]$$

$$+\cos 3\varphi\Big[u_3v_3\left(-2e\sin\theta+\frac{39}{8}e\sin\theta\, e\cos\theta+\frac{1}{8}e\sin\theta\,(e\cos\theta)^2-\frac{11}{8}(e\sin\theta)^3\right)$$

$$+(u_3^{\,2}-v_3^{\,2})\left(\frac{7}{2}+\frac{11}{4}e\cos\theta-\frac{1}{8}(e\cos\theta)^2+\frac{17}{8}(e\sin\theta)^2+\frac{5}{8}(e\cos\theta)^3\right.$$

$$\left.+e\cos\theta\,(e\sin\theta)^2\right)+\left(\frac{3}{2}(u_3^{\,2}+v_3^{\,2})-1\right)\left(\frac{(e\sin\theta)^2}{2}\right.$$

$$\left.-\frac{(e\cos\theta)^2}{2}-\frac{(e\cos\theta)^3}{2}+e\cos\theta\,(e\sin\theta)^2\right)\Big]$$

$$+\cos 4\varphi\Big[-u_3v_3\left(\frac{1}{2}e\sin\theta+\frac{3}{2}e\sin\theta\, e\cos\theta\right)+\frac{7}{8}(u_3^{\,2}-v_3^{\,2})\left(2e\cos\theta+2(e\cos\theta)^2\right.$$

$$\left.-(e\sin\theta)^2\right)\Big]$$

$$+\cos 5\varphi\Big[\frac{-u_3v_3}{8}\left(-21e\cos\theta\, e\sin\theta-26e\sin\theta\,(e\cos\theta)^2+5(e\sin\theta)^3\right)$$

$$+(u_3^{\,2}-v_3^{\,2})\left(\frac{(e\cos\theta)^2}{4}+\frac{(e\cos\theta)^3}{4}-\frac{5}{4}e\cos\theta\,(e\sin\theta)^2-\frac{5}{8}(e\sin\theta)^2\right)\Big]$$

$$+\sin\varphi\Big[-u_3v_3\left(1+\frac{5}{2}e\cos\theta+\frac{3}{2}(e\cos\theta)^2+\frac{5}{2}(e\sin\theta)^2-\frac{3}{2}e\cos\theta\,(e\sin\theta)^2\right)$$

$$+(u_3^{\,2}-v_3^{\,2})\left(e\sin\theta-\frac{1}{2}e\sin\theta\, e\cos\theta-\frac{1}{2}e\sin\theta\,(e\cos\theta)^2-\frac{1}{2}(e\sin\theta)^3\right)$$

(49)

(cont'd on next page)

$$+\left(\frac{3}{2}(u_3^{\,2}+v_3^{\,2})-1\right)\left(e\sin\theta\, e\cos\theta-\frac{1}{4}e\sin\theta\,(e\cos\theta)^2-\frac{3}{4}(e\sin\theta)^3\right)\Big]$$

$$+\sin 2\varphi\Big[-3u_3v_3\left(1-(e\sin\theta)^2-(e\cos\theta)^2-\frac{1}{4}e\sin\theta\; e\cos\theta\right)$$

$$-5(u_3^{\,2}-v_3^{\,2})\left(e\sin\theta+\frac{1}{2}e\sin\theta\, e\cos\theta\right)-(2e\sin\theta+3e\sin\theta\, e\cos\theta)$$

$$\left(\frac{3}{2}(u_3^{\,2}+v_3^{\,2})-1\right)\Big]$$

$$+\sin 3\varphi\Big[u_3v_3\left(7+\frac{11}{2}e\cos\theta+\frac{17}{4}(e\sin\theta)^2-\frac{1}{4}(e\cos\theta)^2+\frac{5}{4}(e\cos\theta)^3\right.$$

$$\left.+\frac{7}{2}e\cos\theta\,(e\sin\theta)^2\right)+(u_3^{\,2}-v_3^{\,2})\left(e\sin\theta+\frac{11}{16}(e\sin\theta)^3-\frac{7}{16}e\sin\theta\,(e\cos\theta)^2\right.$$

$$\left.-\frac{9}{4}e\sin\theta\; e\cos\theta\right)+\left(\frac{3}{2}(u_3^{\,2}+v_3^{\,2})-1\right)\left(-e\sin\theta\; e\cos\theta\right.$$

$$\left.+\frac{1}{4}(e\sin\theta)^3-\frac{5}{4}e\sin\theta\,(e\cos\theta)^2\right)\Big]$$

$$+\sin 4\varphi\Big[u_3v_3\left(6e\cos\theta+6(e\cos\theta)^2-3(e\sin\theta)^2-\frac{3}{4}e\sin\theta\, e\cos\theta\right)$$

$$+(u_3^{\,2}-v_3^{\,2})\left(3e\sin\theta+\frac{9}{2}e\sin\theta\; e\cos\theta\right)\Big]$$

$$+\sin 5\varphi\Big[5u_3v_3\left(\frac{1}{4}(e\cos\theta)^2+\frac{1}{4}(e\cos\theta)^3-\frac{1}{4}(e\sin\theta)^2-\frac{1}{2}e\cos\theta\,(e\sin\theta)^2\right)$$

$$+5(u_3^{\,2}-v_3^{\,2})\left(\frac{1}{4}e\sin\theta\, e\cos\theta+\frac{5}{16}e\sin\theta\,(e\cos\theta)^2-\frac{1}{16}(e\sin\theta)^3\right)\Big]\Big\} \tag{49}$$

V - FIRST ORDER SOLUTION OF THE POLAR OBLATENESS PERTURBATION EQUATIONS

In solving the perturbation equations derived in the preceding section, the general procedure is to solve exactly for as large a piece of the equation as possible, assuming that the parameters appearing in that piece are constant (except that parameter for which one solves in that particular equation).* The remaining terms are then integrated holding all the parameters constant and the results are added to the solutions obtained in the first step. This procedure is justified because it is equivalent to a second application of the variation-of-parameters method in which only first order terms are retained. Thus in the perturbation equation for a parameter x, that part, labeled x_H' in the preceding section, is the piece of the equation that can be solved exactly under the restrictions mentioned above. The remaining part of the equation, which is integrated keeping all the parameters constant, was labeled x_S' in the preceding section.

The equations to be solved to obtain $e \cos \theta$, $e \sin \theta$, u_1, u_2, u_3, v_1, v_2 and v_3 are Eqs. (40 through 47). Of these equations, those lettered "a" may be divided into the sets of coupled equations [(40a) and (41a)], [(44a) and (47a)] and [(42a), (43a), (45a) and (46a)]. These sets are solved by standard methods with the following restriction: In each system of equations, those parameters on the R.H.S. (right-hand side) which do not appear on the L.H.S. are kept constant. Thus, for example, in the system of Eqs. (40a) and (41a) the parameters g, u_3 and v_3 appearing on the R.H.S. are held constant on solving this system.

The first order solutions are then the following:

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* Although the method employed here and that prescribed by the method of averages have different theoretical justifications, the application of the two methods requires the solution of equations which appear to be quite similar.[5]

$$\begin{pmatrix} e\cos\theta \\ e\sin\theta \end{pmatrix} = \cos\sqrt{1+2\epsilon_2}\varphi \begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix} \begin{pmatrix} \dfrac{1-\cos\sqrt{1+2\epsilon_2}\varphi}{1+2\epsilon_2}\epsilon_2 + (e\cos\theta)_0 \\ \dfrac{-\sin\sqrt{1+2\epsilon_2}\varphi}{\sqrt{1+2\epsilon_2}}\epsilon_2 + (e\sin\theta)_0 \end{pmatrix}$$

$$+ \sin\sqrt{1+2\epsilon_2}\varphi \begin{pmatrix} \sin\varphi & \cos\varphi \\ -\cos\varphi & \sin\varphi \end{pmatrix} \begin{pmatrix} \dfrac{1-\cos\sqrt{1+2\epsilon_2}\varphi}{\sqrt{1+2\epsilon_2}}\epsilon_2 + (e\cos\theta)_0\sqrt{1+2\epsilon_2} \\ \dfrac{-\sin\sqrt{1+2\epsilon_2}\varphi}{1+2\epsilon_2}\epsilon_2 + \dfrac{(e\sin\theta)_0}{\sqrt{1+2\epsilon_2}} \end{pmatrix} \tag{50}$$

$$+ \begin{pmatrix} \int (e\cos\theta)_S' \, d\varphi \\ \int (e\sin\theta)_S' \, d\varphi \end{pmatrix}$$

where $\epsilon_2 = \left[\frac{3}{2}(u_{30}^2 + v_{30}^2) - 1\right] \dfrac{3\mu^2 k_2}{g_o^4}$

$$\begin{pmatrix} u_3 \\ v_3 \end{pmatrix} = \begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix} \begin{pmatrix} \cos\sqrt{1+2\epsilon_1}\varphi & \dfrac{\sin\sqrt{1+2\epsilon_1}\varphi}{\sqrt{1+2\epsilon_1}} \\ -\sqrt{1+2\epsilon_1}\sin\sqrt{1+2\epsilon_1}\varphi & \cos\sqrt{1+2\epsilon_1}\varphi \end{pmatrix} \begin{pmatrix} u_{30} \\ v_{30} \end{pmatrix} \tag{51}$$

$$+ \begin{pmatrix} \int (u_3)_S' \, d\varphi \\ \int (v_3)_S' \, d\varphi \end{pmatrix}$$

where $\epsilon_1 = \dfrac{3\mu^2 k_2}{g_o^4} \left(\dfrac{g_3}{g_o}\right)^2$

$$\begin{pmatrix} u_1 \\ u_2 \\ v_1 \\ v_2 \end{pmatrix} = \frac{1}{4\sqrt{\alpha^2+2\beta+1}} \begin{pmatrix} a_c & a_s & b_s & -b_c \\ -a_s & +a_c & b_c & b_s \\ c_s & -c_c & d_c & d_s \\ c_c & c_s & -d_s & d_c \end{pmatrix} \begin{pmatrix} u_{10} \\ u_{20} \\ v_{10} \\ v_{20} \end{pmatrix} + \begin{pmatrix} \int (u_1)_S' \, d\varphi \\ \int (u_2)_S' \, d\varphi \\ \int (v_1)_S' \, d\varphi \\ \int (v_2)_S' \, d\varphi \end{pmatrix} \tag{52}$$

where

$$a_c = -(|\lambda_2| + 2\beta) \cos |\lambda_1|\varphi + (|\lambda_1| + 2\beta) \cos |\lambda_2|\varphi - (|\lambda_4| - 2\beta) \cos |\lambda_3|\varphi + (|\lambda_3| - 2\beta) \cos |\lambda_4|\varphi$$

$$a_s = -(|\lambda_2| + 2\beta) \sin |\lambda_1|\varphi + (|\lambda_1| + 2\beta) \sin |\lambda_2|\varphi - (|\lambda_4| - 2\beta) \sin |\lambda_3|\varphi + (|\lambda_3| - 2\beta) \sin |\lambda_4|\varphi$$

$$b_s = (|\lambda_2| - 2\alpha) \sin |\lambda_1|\varphi - (|\lambda_1| - 2\alpha) \sin |\lambda_2|\varphi - (|\lambda_4| - 2\alpha) \sin |\lambda_3|\varphi + (|\lambda_3| - 2\alpha) \sin |\lambda_4|\varphi$$

$$b_c = (|\lambda_2| - 2\alpha) \cos |\lambda_1|\varphi - (|\lambda_1| - 2\alpha) \cos |\lambda_2|\varphi - (|\lambda_4| - 2\alpha) \cos |\lambda_3|\varphi + (|\lambda_3| - 2\alpha) \cos |\lambda_4|\varphi \tag{53}$$

(cont'd on next page)

$$c_c = -(|\lambda_2| + 2\beta)\cos|\lambda_1|\varphi + (|\lambda_1| + 2\beta)\cos|\lambda_2|\varphi + (|\lambda_4| - 2\beta)\cos|\lambda_3|\varphi$$
$$- (|\lambda_3| - 2\beta)\cos|\lambda_4|\varphi$$

$$c_s = -(|\lambda_2| + 2\beta)\sin|\lambda_1|\varphi + (|\lambda_1| + 2\beta)\sin|\lambda_2|\varphi + (|\lambda_4| - 2\beta)\sin|\lambda_3|\varphi$$
$$- (|\lambda_3| - 2\beta)\sin|\lambda_4|\varphi \qquad \text{(53) cont'd}$$

$$d_c = -(|\lambda_2| - 2\alpha)\cos|\lambda_1|\varphi + (|\lambda_1| - 2\alpha)\cos|\lambda_2|\varphi - (|\lambda_4| - 2\alpha)\cos|\lambda_3|\varphi$$
$$+ (|\lambda_3| - 2\alpha)\cos|\lambda_4|\varphi$$

$$d_s = -(|\lambda_2| - 2\alpha)\sin|\lambda_1|\varphi + (|\lambda_1| - 2\alpha)\sin|\lambda_2|\varphi - (|\lambda_4| - 2\alpha)\sin|\lambda_3|\varphi$$
$$+ (|\lambda_3| - 2\alpha)\sin|\lambda_4|\varphi$$

and

$$\alpha = \epsilon\frac{g_3}{g_o}$$

$$\beta = \epsilon\left(\frac{g_3}{g_o}\right)^2$$

$$|\lambda_1| = \alpha + 1 + \sqrt{\alpha^2 + 2\beta + 1}$$

$$|\lambda_2| = \alpha + 1 - \sqrt{\alpha^2 + 2\beta + 1}$$

$$|\lambda_3| = (\alpha - 1) + \sqrt{\alpha^2 + 2\beta + 1}$$

$$|\lambda_4| = (\alpha - 1) - \sqrt{\alpha^2 + 2\beta + 1}$$

The absolute value signs used here indicate that a factor i is omitted from the λ's which are the characteristic roots for the system of Eqs. (42a, 43a, 45a, 46a).

The equations to be solved to obtain g are Eqs. (48a) and (48b). The R.H.S. of Eq. (48a) is a perfect differential if one takes into account the equations for u_3' and v_3' (Eqs. 44a, 44b, 47a, 47b). In Eq. (48b) the parameters appearing in the R.H.S. are given their initial values and one obtains finally the following expression for g^4

$$g^4 = g_o^4 - 12\,\mu^2 k_2 \left[\left(u_3 \cos\varphi + v_3 \sin\varphi\right)^2 - \left(u_{30}\right)^2\right] + 4\int g_S{}'\, d\varphi \qquad (54)$$

The equation to be solved to obtain t_0 is Eq. (49). The L.H.S. of this equation is a perfect differential, provided one assumes that a is a constant, a_0, and that the t_0 occurring in the second term on the L.H.S. is also a constant. In a first order solution this is justified. Thus, the L.H.S. of the equation integrates to

$$t_0\,(t) + \frac{3a_0}{\mu}\,\Phi\,(t - t_{00})$$

In the R.H.S. of the equation, the parameters are assumed to have their initial values and the integration is performed with respect to φ. Thus

$$t_0\,(t) = t_{00} - \frac{3a_0}{\mu}\Phi(t-t_{00}) - \frac{3k_2}{\left[1+(e\cos\theta)_0\right]^2 g_o}\left[\frac{3}{2}\left(u_{30}{}^2 + v_{30}{}^2\right) - 1\right]\varphi \qquad (55)$$

+ integral of other terms.

The limits of integration are zero and φ.

SECTION VI - DISCUSSION OF THE RESULTS AND APPLICATIONS

Now that the perturbation equations and their first order solutions are available for examination, some distinctive features of the parameters become apparent. It has already been noted that the parameters U and V are perpendicular unit vectors which are to be regarded as rigidly attached to the angular momentum vector G throughout the motion. These parameters thus differ in an essential way from any of the conventional sets of parameters such as the Delaunay elements or initial conditions, because to relate the initial values of the parameters with their values at time t requires knowledge not only of the position and velocity initially and at time t, but also a knowledge of the trajectory between these times. For any conventional set of elements, on the other hand, knowledge of the initial and terminal conditions is sufficient to determine the initial and terminal values of the elements. It may thus appear at first sight that the elements used in this report involve complications that are not present in the use of conventional elements. It must, however, also be recalled that the present elements are so defined that the independent variable φ has no perturbation derivative, while with conventional elements the independent variable, usually either true or mean anomaly, does have a perturbation derivative, which introduces complications in the derivation, and integration of the perturbation equations. Further, even though the present elements are functions of the trajectory (and hence of the particular perturbing function used), once the perturbation equations have been integrated the fact that the solution of these equations must be used to determine the elements poses no fundamental problem.

In this report only the first order solution of the perturbation equations has been presented. To obtain the second order solution Eqs. (40) to (49) are integrated again replacing those parameters held constant in the first order integration by their first order solutions. The integration of these equations involves a great deal of routine trigonometric manipulation and will be the subject of a later report. It is, however, possible to state a general conclusion on the results of the integration. This conclusion is that the second order terms will be small compared to the first order terms for a time of the order of 100 periods. This means that for any problem (for which the first order solution has sufficient precision) the first order solution is usable and valid for about 100 periods. The reason for this is that in the second order solution, terms of the form

$$\frac{\sin \epsilon \varphi}{\epsilon} \quad \text{and} \quad \frac{1 - \cos \epsilon \varphi}{\epsilon} \tag{56}$$

occur with coefficients of the form $A k_2^2$ where A is of order unity. Noting that, for any ϵ

$$\left| \frac{1 - \cos \epsilon \varphi}{\epsilon} \right| = \left| \frac{1}{2} \frac{\sin^2 \epsilon \varphi}{\epsilon} \right| \leq \left| \frac{\sin \epsilon \varphi}{\epsilon} \right| \leq \varphi$$

it is evident that no such term can creep into first order so long as

$$A k_2^2 \varphi < k_2$$

or

$$\frac{\varphi}{2\pi} < \frac{1}{2\pi A k_2} \sim 100$$

It should be remarked that if ϵ vanishes the first of the terms (56) is secular and the second is a constant. It turns out that for at least two particular sets of initial conditions there will be secular terms, namely, for initial conditions such that the angle of inclination is 63.4° and 67.8°. Thus, critical angles of inclination occur in this formulation, but not in the same way as in conventional theories, for which only one critical angle has been found. The significant difference is that in conventional theories the critical angle appears as a singularity in the second order solution, whereas in the present theory the second order solution has no singularity, and while it is unbounded in time, it will not affect the first order solution for about 100 periods.

One might inquire what sort of precision can be expected from the first order theory. In order to discuss this question, it must first be remarked that parameters associated with the Kepler problem may be separated into two categories. Parameters such as the semimajor axis, the eccentricity, and the angle of inclination, as well as functions of such parameters have only short period terms in their first order corrections. Other parameters such as argument of perigee conventionally contain not only short period terms but also secular terms. No first order secular terms appear in this formulation because of the way in which the differential equations (40) to (47) are separated. The closed form contribution to the first order solutions obtained in Section IV from Eqs. (40a), (41a),...(47a) include the analogues to the secular terms as well as such short period terms as could be included in the closed form integration. Suppose now that one numerically compares trajectory predictions based on the Kepler problem, based on the first order solution derived in this report and based on a high precision numerical integration. If the comparison is made for $\varphi = 2\pi$, all short period terms will disappear. Those parameters involving only short period terms should be the same for both the Kepler and the first order predictions and should agree to about six significant digits (since $k_2^{\,2} = 10^{-6}$) with the precision calculation. The remaining parameters should be given to about three more significant digits by the first order theory than by the Kepler estimate. If the comparison were made on functions of the elements, rather than on the elements themselves, one would still expect the first order theory to yield about three more significant digits than the Kepler estimate except for functions independent of parameters containing secular terms. A comparison, at $\varphi = 2\pi$, on position and velocity would thus be expected to yield, in general, three more digits from the first order estimate than for the Kepler estimate. Preliminary numerical comparisons indicate that this is indeed the case.

The application of the theory developed in this report for prediction is fairly direct. To obtain position, velocity and time corresponding to a specified φ, one simply evaluates the elements from Eqs. (50) - (55), and then substitutes in Eqs. (5), (6) and (15). To obtain position and velocity at a specified time it is necessary to replace all elements in Eq. (15) except t and φ by their expressions in terms of φ, to obtain a transcendental relation between φ and t. The angle φ would then be obtained by numerical solution of this equation. Once φ is known, position and velocity are obtained as above.

The boundary value problem is somewhat more difficult. In this case one would require knowledge of seven conditions, some given at the initial point and the rest at the terminal point. Now Eqs. (50) to (55) give the parameters as functions of φ and Eq. (15) relates φ and t. Eqs. (5) and (6) give position and velocity as functions of the parameters. The boundary conditions would thus give seven equations for the determination of six independent parameters and t. The solution of these equations would have to be carried out numerically because of their transcendental character.

In conclusion, one might comment on some special solutions of the perturbation equations. If the initial conditions are such that the initial orbit is either equatorial or polar the U and V vectors are constants of the motion. The perturbation equations (42) - (47) for $\dot{U}$ and $\dot{V}$ contain $\frac{g_3}{G}$ as a factor on the right hand side. For polar orbits g_3 vanishes and hence U and V are constant vectors. For an equatorial orbit $u_3 = v_3 = 0$ and hence, again, the right hand side of Eqs. (42) - (47) for $\dot{U}$ and $\dot{V}$ vanishes. This last result illustrates one advantage of these parameters. The conventional elements include longitude of the node and argument of perigee which are not defined for equatorial orbits and hence modifications are required for the treatment of this case.

In Eqs. (40) and (41) the expression $(1 - 3\cos^2 i)$ can be shown to be a factor of the right hand side. This factor vanishes for an angle of inclination of 54.74° and hence for an orbit initially at this inclination the eccentricity and the parameter θ (angle between U and perigee) are constants of the motion. The critical angles 63.4° and 67.8° which appear in the second order theory have no obvious significance for the parameters used in this report. It is curious, however that these three angles have the property that

$$\cos^2 i = \frac{1}{3}, \frac{1}{5}, \frac{1}{7}$$

respectively.

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HAYES INTERNATIONAL CORPORATION

AN EMPIRICAL STUDY OF CONFIDENCE LIMITS FOR DESIRED CUTOFF CONDITIONS

By

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SUMMARY

A statistical procedure, taking into consideration variations due to changes in launch times as well as errors introduced by the path adaptive guidance polynomials, was designed to obtain confidence limits for desired cutoff conditions such as radius, velocity, etc. The following upper bounds for 2σ limits were obtained for a given example: 2100 meters for the radius, 1.615 meters/second for the velocity, .2692 degrees for the flight path angle, and .026 degrees for the orbital inclination.

INTRODUCTION

It is our purpose in this study to design and perform an experiment that can be used to obtain confidence limits for desired cutoff conditions such as radius, flight path angle, velocity, and orbital inclination. We are interested in two results - the design of a statistical procedure that could be employed for similar problems and the actual numerical results from this particular experiment. The confidence limits obtained will be such that they will take into consideration variations in launch times across a selected launch window as well as errors introduced by the path adaptive guidance polynomials.

In the example that we considered, a volume of trajectories was computed by the Boeing Company using the theory of calculus of variations. Each of the trajectories obtained, if flown, would place a vehicle at the desired end conditions in an optimum manner. Likewise, multidimensional polynomials were computed by the Boeing Company, to fit or approximate the volume of optimum trajectories. In our experiment the data consists of results obtained by running on a computer actual trajectory simulations with guidance commands being provided by the polynomials. These simulations, as run by the Boeing Company, used the steering and cutoff polynomials to guide the flight.

THE DESIGN OF THE EXPERIMENT

Now in the design of our experiment, instead of selecting a random sample from the volume of all optimum trajectories associated with a given cutoff condition, we elected to take a sample from a sub-set of the universe of all optimum trajectories.

A description of the sub-set of the universe from which we selected our sample involves the definition of a nominal trajectory. In the process of generating a volume of optimum trajectories a certain optimum trajectory that satisfies selected performance criteria is classified as "the nominal trajectory". The universe of trajectories that we considered in our experiment was generated by considering variations or perturbations of parameters from their values given for the nominal trajectory.

Thus each vehicle parameter and flight parameter was assumed to have a nominal value. In a like manner the tolerance or standard deviation measuring variation from this nominal value was taken to be known for each parameter. It was further supposed that deviation from a nominal trajectory could be caused by any one or combination of a number of independent error sources or parameters, each of which was normally distributed about the nominal value as a mean.

Under these assumptions the universe that we considered consisted of trajectories, $T(x_1', x_2', x_3', \ldots x_n')$ such that $\prod_{i=1}^{n}\left[1-P(x_i''<x_i<x_i')\right]$ $= .05$ where x_i' and x_i'' are perturbed values of x_i $(i = 1, 2, \ldots, n)$ such that $x_i' - \mu x_i = \mu x_i - x_i''$. This universe or family of trajectories we called "5 percent level trajectories".

For example if a trajectory is to be obtained by the deviation of one parameter from a nominal value $1-P(x_1''<x_1<x_1') = .05$ or $P(x_1''<x_1<x_1') = .95$. Thus $x_1' = \mu + 1.96\sigma$ and $x_1'' = \mu - 1.96\sigma$ or x_1' is the sum of the nominal value of the variable and approximately two standard deviations of the variable.

If a trajectory is to be obtained by the deviation of <u>two</u> parameters then $\left[1-P(-x_1''<x_1<x_1')\right]\left[1-P(-x_2''<x_2<x_2')\right] = .05$. An infinite number of combinations of x_1' and x_2' could be assumed to give these trajectories. For example if $\overline{x}_1$ and $\overline{x}_2$ are nominal values then five possible combinations are:

$$\overline{x}_1 + .67\sigma_{x_1},\ \overline{x}_2 + 1.65\sigma_{x_2}$$

$$\overline{x}_1 + \sigma_{x_1},\ \overline{x}_2 + 1.42\sigma_{x_2}$$

$$\overline{x}_1 + 1.22\sigma_{x_1},\ \overline{x}_2 + 1.22\sigma_{x_2}$$

$$\overline{x}_1 + 1.42\sigma_{x_1},\ \overline{x}_2 + \sigma_{x_2}$$

$$\overline{x}_1 + 1.65\sigma_{x_1},\ \overline{x}_2 + .67\sigma_{x_2}$$

THE VARIANCE OF CUTOFF ERRORS

Now let us define y to be a variable representing any one of the cutoff conditions: radius, flight path angle, velocity, or orbital inclination. Then by the variance of y, σ_y^2, we mean the average of the squares of the deviations of y from the desired cutoff condition.

We observe that, on the average, trajectories formed by small deviations of parameters from the nominal, give cutoff errors close to those given by the nominal. Since variations in a parameter are considered to be normally distributed about the nominal value as a mean, a large percentage of the trajectories will have cutoff errors about the same size as those for the nominal. A comparison of a sample of 5 percent level trajectories with the nominal clearly indicates that the variance of any cutoff condition for the 5 percent level trajectories will be greater than the variance of the universe of all optimum trajectories.

The next step in our experiment consisted of selecting a sample of forty-six (46) trajectories from the universe of all 5 percent level trajectories. Eleven (11) of these trajectories were generated by deviating one parameter from the nominal. The eleven parameters were chosen because they seemed to produce the largest cutoff errors. The parameters selected to generate these off-nominal trajectories were: stage 1 thrust, specific impulse, and inertia weight; stage 2 thrust, specific impulse and inertia weight; stage 3 thrust, specific impulse and inertia weight along with head wind and left cross wind. Of these eleven parameters the ones that produced the largest errors were then combined using the probability theory indicated previously. Fifteen (15) trajectories were generated by deviating two parameters from their nominals such as stage 1 thrust along with stage 2 inertia weight. Twenty (20) of the trajectories were generated by deviating simultaneously three parameters from their nominals. As indicated a special effort was made in the selection of the parameters and combination of parameters to select those that would make as large as possible the errors in cutoff conditions. For example in a combination of parameters the direction of the variations were selected so that the resulting errors would be in the same direction. To summarize, the forty-six (46) trajectories used in the sample were selected by the Saturn Booster Branch of the Boeing Company to be 5 percent level trajectories that would produce the largest errors in cutoff conditions.

Thus

$$\sigma_y^2 \text{ (Sample)} > \sigma_y^2 \text{ (5 Percent Level)}$$

$$\sigma_y^2 \text{ (5 Percent Level)} > \sigma_y^2 \text{ (Universe)}$$

$$\sigma_y^2 \text{ (Sample)} > \sigma_y^2 \text{ (Universe)}$$

This selection technique as described produces a sample which will have a variance that can serve as an upper bound for the variance of the universe of all trajectories generated by allowing parameters to assume off-nominal values. We could call this " a 95 percent level upper bound" since the probability that parameters will deviate from the nominal by more than the parameters used to obtain this upper bound in less than .05.

However, due to the cost involved it was decided that the size of the sample was too large. From a careful study of the data a more select sample of size 10 was chosen. Once again the parameters that were chosen for this sample were selected because they induced the largest errors in cutoff conditions. A comparison of the variance of the sample of size 10 with the variance of the sample of size 46 is given in the following table:

Cutoff Condition	S_1^2 (Size 10)	S_2^2 (Size 46)	$F = \frac{S_1^2}{S_2^2}$	F(97.5%)
Radius	1,833,578	593,310	3.09	2.43
Velocity	.938	.3462	2.72	2.43
Flight Path Angle	.02566	.0083	3.08	2.43
Orbital Inclination	.0000053	.0000166	.32	2.43

Under the assumption that the two samples come from populations with equal variance there is less than a 2.5 percent chance of getting a variance in a sample of size 10 that deviates as much from the sample of size 46 as the sample selected. This is true for three cutoff conditions: radius, velocity, and flight path angle. Hence, it can be stated that the variance of the selected sample of size 10 is an upper bound for the variance of the universe of all trajectories in the volume relative to these cutoff conditions. It is probably true that the selected sample of size 10 produces an upper bound for the variance of the variable, orbital inclination.

The ten optimum trajectories that were selected because they seemed to produce extremely large cutoff errors were generated by: (1) 2σ head wind, (2) 2σ stage two thrust, (3) 2σ stage two inertia weight, (4) -1.2σ stage two thrust along with 1.2σ stage two inertia weight, (5) 1.2σ head wind along with 1.2σ stage two inertia weight, (6) 1.2σ stage two inertia weight combined with 1.2σ stage three inertia weight, (7) 1.2σ head wind and a -1.2σ stage two thrust, (8) $.9\sigma$ head wind, $-.9\sigma$ stage two thrust, and $.9\sigma$ stage two inertia weight, (9) $.9\sigma$ head wind, $.9\sigma$ stage two inertia weight, and a $.9\sigma$ stage three inertia weight, (10) $-.9\sigma$ stage two thrust, $.9\sigma$ stage two inertia weight and a $.9\sigma$ stage three inertia weight.

VARIATIONS IN LAUNCH TIME

Note that in all the preceding discussion, we have considered a fixed launch time. Let us now extend our sampling process to the entire launch window. We will design our procedure to involve running simulated flights at seven different launch times in the launch window; i.e., -30, -20, -10, 0, 10, 20, 30 minutes. The results of the data obtained from this experiment will be used to construct upper bounds for confidence limits for the errors of a given cutoff condition. The confidence limits obtained will cover the complete launch window and include all possible trajectories that would be included by chance 95 percent of the time.

The variance of the seven samples taken at the seven different launch times will be pooled or averaged together to give an upper bound for the variance of the universe of all optimum trajectories throughout the launch window. We make use of the following formula for this analysis.

$$s_7^2 = \frac{(n_1 - 1)\, s_1^2 + (n_2 - 1)\, s_2^2 + \ldots (n_7 -)\, s_7^2}{n_1 + n_2 + \ldots + n_7 - 7}$$

Using s_7^2 as an upper bound for the variance of the universe of all optimum trajectories, the following upper bounds are obtained for 2σ.

Cutoff Condition	Upper Bound for 2σ
Radius	2656 meters
Velocity	1.8544 meters/sec
Flight Path Angle	.3142 degrees
Orbital Inclination	.0458 degrees

The process by which we selected our samples indicates that a somewhat smaller upper bound can be obtained by using statistical theory utilizing the range. At the same time it is evident that the range of our selected sample of size ten at a given launch time is undoubtedly an upper bound for the range of random sample of size ten from the universe of all trajectories at a given launch time.

Thus, let us assume that we are taking random samples of size 10 from a universe made up of optimum trajectories generated by allowing the vehicle and flight parameters to vary. Since our samples are small (of size 10) the range and standard deviation of a sample are likely to fluctuate together. Thus the range may be used to estimate variance with little loss of efficiency. Once again, assuming that the errors for each variable are normally distributed, we utilize tabulated tables for the w distribution where $w = R/\sigma$.

To estimate the standard deviation of the universe we calculate the average range of the 7 samples. Call this value, $\overline{R}$. For samples of size 10 the expected value of w is 3.078. Thus an estimate of the standard deviation of the universe is given as $\frac{\overline{R}}{3.078}$. In other words to take $\overline{R}$ to be an estimate of the mean value of the range of all samples of size 10 is the equivalent of taking the standard deviation of the universe to be $\overline{R}/3.078$. Since the range for each of the selected samples at different launch times is an upper bound for the range of random samples at these times, then

$$E(\sigma) = \frac{\overline{R}}{3.078} \text{ (Random Samples)} < \frac{\overline{R}}{3.078} \text{ (Selected Samples)}$$

Thus $\overline{R}/3.078$ for our selected samples gives an upper bound for σ of the universe.

Cutoff Condition	Upper Bound for 2σ
Radius	2100 meters
Velocity	1.615 meters/sec
Flight Path Angle	.2692 degrees
Orbital Inclination	.026 degrees

VARIABILITY TESTS

In the preceding theory we have made the basic assumption that the variability of output errors remains constant during changes of launch window. In particular we have made this assumption for the universe from which we picked out "selected sample". We wish now to test whether or not the variability of output errors remains constant across the launch window. As a word of caution it should be remembered that if the variation as indicated by the samples should prove to be significant it could be due to departure from normality within the groups rather than departure from heterogeneity.

Two tests will be made for each variable. The $\frac{\max s_i^2}{\min s_i^2}$ will afford a quick test for comparing the variance estimates. Cochran's Tests for the homogeneity of variance $\frac{\max s_i^2}{\Sigma s_i^2}$ tests whether one variance is significantly larger than the others.

Variable	$\frac{\text{Max } S_i^2}{\text{Min } S_i^2}$	5% Point	Result	$\frac{\text{Max } S_i^2}{\Sigma S_i^2}$	5% Point	Result
r	1.06	7.42	not significant	.15	.315	not significant
θ	1.59	7.42	not significant	.156	.315	not significant
V	1.17	7.42	not significant	.159	.315	not significant
i	16.2	7.42	significant	.54	.315	significant

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HAYES INTERNATIONAL CORPORATION

DERIVATION OF A GUIDANCE FUNCTION MODEL

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SUMMARY

This report presents the derivation of a mathematical model for fitting the steering function. The solution gives cot θ ($\theta = \psi + \phi$ where ψ is the steering angle and ϕ = arc tan $\frac{y}{x}$) as a function of time and state variables along the trajectory. This function evaluated at $t = t_o$ should be the desired steering function. No end conditions were considered as all constants of integration were combined with unknown constants in the power series expansion; it is proposed that curve fitting techniques will be used to obtain these constants.

INTRODUCTION

This paper is concerned with the development of a mathematical model that may be used as a guidance function. The basic requirement of a guidance function is that it instantly converts sensed state variables of the vehicle into command signals to enable the vehicle to follow a newly selected optimal path.

At this time multivariate polynomials are being used to express the guidance parameters in terms of the state and performance variables of the vehicle. From all reports these polynomials seem to be adequate to represent the problem encountered at this time. However, at several meetings of those concerned with this phase of the guidance problem, opinions have been expressed that some other type function might better represent the relationship between guidance parameters and state variables.

So this research was motivated by the problem of trying to develop, if possible, a function or form of a function that would represent the relationship between the state variables and guidance parameters. The model that will be obtained will have three important properties. First of all its form will not be assumed in any way. Secondly, the functional relationship will be developed from the equations that define the motion and the conditions that insure an optimum trajectory. Finally, the relationship will contain a number of undetermined coefficients which will need to be obtained by some method of curve fitting techniques.

THE PROBLEM CONCEPT

In this paper a mathematical model defining the steering function in terms of instantaneous state variables is developed for the powered flight problem defined as follows:

1. Motion is assumed to occur in a vacuum.
2. Only two dimensional motion is considered.
3. Rigid body dynamics is neglected.
4. The earth is assumed to be spherical and homogeneous.
5. A constant applied force (F) is considered.
6. The time rate of change of the mass ($\dot{m}$) of the vehicle is constant.

Langrangian Multipliers are used to formulate necessary conditions for extremizing some variable such as propellant consumption or burning time.

Equations Defining the Problem

The differential equations which define the motion of the vehicle may be written as:

$$\begin{aligned} \ddot{x} &= \frac{F}{m} \sin \psi - \frac{kx}{r^3} \\ \ddot{y} &= \frac{F}{m} \cos \psi - \frac{ky}{r^3} \end{aligned} \tag{1}$$

The coordinate system, x, y is chosen so that x is parallel to the surface of the earth, and y is perpendicular to the surface. The dot represents differentiation with respect to time. F is the thrust magnitude which we assume to be constant. Likewise, k is considered as a given constant. The mass, m, is of the form $m_o + \dot{m}_o t$ where m_o and $\dot{m}_o$ are considered as constants. The control variable is ψ, the direction of the thrust vector measured positive from the upward vertical.

Now consider the change of variables defined by

$$\begin{aligned} x &= r\cos\phi & x_1 &= \dot{r} & x_3 &= r \\ y &= r\sin\phi & x_2 &= r\dot{\phi} & \theta &= \psi + \phi \end{aligned} \tag{2}$$

Under this transformation equations (1) become

$$\begin{aligned} \dot{x}_1 &= \frac{x_2^2}{x_3} - \frac{k}{x_3^2} + \frac{F}{m}\sin\theta \\ \dot{x}_2 &= -\frac{x_1 x_2}{x_3} + \frac{F}{m}\cos\theta \end{aligned} \tag{3}$$

The function whose time integral is to be extremized may be defined as $G = 1 + \Sigma_i \lambda_i g_i$ where λ_i are the undetermined Lagrangian Multipliers, and

$$\begin{aligned} g_1 &= \dot{x}_1 - \frac{x_2^2}{x_3} + \frac{k}{x_3^2} - \frac{F}{m}\sin\theta = 0 \\ g_2 &= \dot{x}_2 + \frac{x_1 x_2}{x_3} - \frac{F}{m}\cos\theta = 0 \\ g_3 &= \dot{x}_3 - x_1 = 0 \\ g_4 &= \dot{x}_4 - \frac{x_2}{x_3} = 0 \\ g_5 &= \dot{m} - \dot{m}_0 = 0 \end{aligned} \tag{4}$$

Applying the Euler-Lagrange conditions to the function G results in the following system of equations:

$$\dot{x}_1 = \frac{x_2^2}{x_3} - \frac{k}{x_3^2} + \frac{F}{m} \sin\theta \tag{5}$$

$$\dot{x}_2 = \frac{-x_1 x_2}{x_3} + \frac{F}{m} \cos\theta \tag{6}$$

$$\dot{\lambda}_1 = \frac{x_2}{x_3} \lambda_2 - \lambda_3 \tag{7}$$

$$\dot{\lambda}_2 = \frac{-2x_2}{x_3} \lambda_1 + \frac{x_1}{x_3} \lambda_2 - \frac{1}{x_3} \lambda_4 \tag{8}$$

$$\dot{\lambda}_3 = \left[\left(\frac{x_2}{x_3}\right)^2 - \frac{2k}{x_3^3}\right] \lambda_1 - \frac{x_1 x_2}{x_3^2} \lambda_2 + \frac{x_2}{x_3^2} \lambda_4 \tag{9}$$

$$\dot{\lambda}_4 = 0 \tag{10}$$

$$\dot{\lambda}_5 = \frac{F}{m^2} \sin\theta \, \lambda_1 + \frac{F}{m^2} \cos\theta \, \lambda_2 \tag{11}$$

$$-\frac{F}{m} \cos\theta \, \lambda_1 + \frac{F}{m} \sin\theta \, \lambda_2 = 0 \tag{12}$$

Since G is explicitly independent of the independent variable, t, a first integral of the system can be shown to be

$$\lambda_1 \dot{x}_1 + \lambda_2 \dot{x}_2 + \lambda_3 \dot{x}_3 + \lambda_4 \dot{x}_4 + \lambda_5 \dot{m} = c_1 \tag{13}$$

By substituting in equation (11) the values given for $\frac{F}{m} \sin\theta$ and $\frac{F}{m} \cos\theta$ in equations (5) and (6) the following relationship is obtained:

$$\lambda_1 \dot{x}_1 + \lambda_2 \dot{x}_2 - \frac{x_2^2}{x_3} \lambda_1 + \frac{k}{x_3^2} \lambda_1 + \frac{x_1 x_2}{x_3} \lambda_2 = m \dot{\lambda}_5 \tag{14}$$

By eliminating $\lambda_1 \dot{x}_1 + \lambda_2 \dot{x}_2$ between equations (13) and (14) the result may be expressed as

$$\frac{x_2^2}{x_3} \lambda_1 - \frac{k}{x_3^2} \lambda_1 - \frac{x_1 x_2 \lambda_2}{x_3} + \dot{x}_1 \lambda_3 + \dot{x}_4 \lambda_4 + \dot{m}\lambda_5 + m\dot{\lambda}_5 = c_1 \tag{15}$$

Now multiply (7) by x_1, (8) by x_2 and (9) by $2x_3$, and add to obtain

$$x_1 \dot{\lambda}_1 + x_2 \dot{\lambda}_2 + 2x_3 \dot{\lambda}_3 = - x_1 \lambda_3 - \frac{4k}{x_3^2} \lambda_1 + \frac{x_2}{x_3} \lambda_4 \quad \text{and} \tag{16}$$

integrate both sides to obtain:

$$x_1 \lambda_1 + x_2 \lambda_2 + 2x_3 \lambda_3 = \int \Big[\dot{x}_1 \lambda_1 + \dot{x}_2 \lambda_2 + 2\dot{x}_3 \lambda_3 - x_1 \lambda_3 - \frac{4k}{x_3^2} \lambda_1 + \frac{x_2}{x_3} \lambda_4 \Big] dt + c_3 \tag{17}$$

This simplifies (by using 13) to

$$x_1 \lambda_1 + x_2 \lambda_2 + 2x_3 \lambda_3 = -\int \frac{4k}{x_3^2} \lambda_1 \, dt - \int \lambda_5 \dot{m} \, dt + c_1 t + c_3 \tag{18}$$

The limits of integration throughout this development are from t to t_c. The values of all state variables at t_c are lumped together as one arbitrary constant.

Now solve $x_1 \lambda_3$ in equation (15) and add this result to $x_3 \dot{\lambda}_3$ as

found in equation (9) to obtain:

$$x_3 \dot{\lambda}_3 + \dot{x}_3 \lambda_3 + m\dot{\lambda}_5 + \dot{m}\lambda_5 = c_1 - \frac{k}{x_3^2} \lambda_1 \tag{19}$$

which after integration with respect to t becomes

$$x_3 \lambda_3 + m\lambda_5 = -\int \frac{k}{x_3^2} \lambda_1 \, dt + c_1 t + c_4 \tag{20}$$

Substituting the result obtained for $\int \frac{k}{x_3^2} \lambda_1 \, dt$ in (20) into equation (18)

yields

$$x_1 \lambda_1 + x_2 \lambda_2 - 2x_3 \lambda_3 = 4m\lambda_5 - 3c_1 t + c_5 - \int \dot{m}\lambda_5 \, dt \tag{21}$$

Now replace $x_3 \lambda_3$ with the value given in equation (7); simplify and

integrate to obtain

$$x_3 \lambda_1 - \int x_3 \lambda_3 \, dt = \int 4 m \lambda_5 \, dt - \iint \dot{m} \lambda_5 \, dt - {}^3/_2 \, c_1 t^2 + c_5 t + c_6 \tag{22}$$

Integrate $\int x_3 \lambda_3 \, dt$ by parts to obtain using equation (20)

$$\int x_3 \lambda_3 \, dt = x_3 \lambda_3 \, t + \int \frac{kt}{x_3^2} \lambda_1 \, dt - \frac{c_1 t^2}{2} + m \lambda_5 \, t - \int (m\lambda_5) dt \tag{23}$$

Now apply the mean value theorem for integrals to the integral $\int \frac{kt \lambda_1 \, dt}{x_3^2}$.
Write $\int_t^{t_c} \frac{kt \lambda_1 \, dt}{x_3^2}$ as $\int_t^{t_p} \frac{kt \lambda_1 \, dt}{x_3^2} + c_7$ where the interval from t to t_p is picked so that $\frac{k\lambda_1}{x_3^2}$ does not change sign in the interval $[t, t_p]$. Then $\int_t^{t_c} \frac{kt \lambda_1 \, dt}{x_3^2}$ can be written as $a \int_t^{t_p} \frac{k \lambda_1 \, dt}{x_3^2} + c_7 = a \int_t^{t_c} \frac{k \lambda_1 \, dt}{x_3^2} + c_8$ where $c_8 = c_7 - a \int_{t_p}^{t_c} \frac{k \lambda_1 \, dt}{x_3^2}$ and $t \leq a \leq t_p$. Now replace $\int \frac{k \lambda_1 \, dt}{x_3^2}$ by its value in equation (20). Then $\int \frac{kt \lambda_1 \, dt}{x_3^2} = - a x_3 \lambda_3 - a m \lambda_5 + a c_1 t + c_9$ (24)

Substituting this result in equation (23) and then in (22) yields

$$x_3 \lambda_1 + (a-t) x_3 \lambda_3 = \int 3m\lambda_5 \, dt - \iint \dot{m} \lambda_5 \, dt + m\lambda_5 \, (c_{10} - t) - c_{11} t^2 + c_{12} t + c_{13} \tag{25}$$

Equations (15), (21) and (25) can be so arranged that the right side of the equations are functions of m, t, λ_5, and $\dot{\lambda}_5$. Since $t = \frac{m - m_0}{\dot{m}}$ the right sides can be considered as functions of m, λ_5, $\dot{\lambda}_5$. Now remember, in this study we are not attempting to solve these equations but to find a model with undetermined coefficients that will satisfy the equations. Thus to find such a model we now assume that the right side of each equation can be expressed as a power series in $\left(\frac{F}{m}\right)$ to give

$$\left(\frac{x_2^2}{x_3} - \frac{k}{x_3^2}\right) \lambda_1 - \frac{x_1 x_2}{x_3} \lambda_2 + x_1 \lambda_3 + c \frac{x_2}{x_3} = \sum_{i=0}^{\infty} b_i \left(\frac{F}{m}\right)^i \tag{26}$$

$$x_1 \lambda_1 + x_2 \lambda_2 - 2 x_3 \lambda_3 = \sum_{i=0}^{\infty} c_i \left(\frac{F}{m}\right)^i ; \quad x_3 \lambda_1 + x_3 (a-t) \lambda_3 = \sum_{i=0}^{\infty} d_i \left(\frac{F}{m}\right)^i$$

The b_i, c_i and d_i are unknown coefficients, which when they appear in the model, will need to be evaluated by some curve fitting technique that fits the model to the space of optimum trajectories. Now considering these equations as three equations in unknowns λ_1, λ_2, λ_3, we solve for λ_1, and λ_2. Since $\cot\theta = \frac{\lambda_2}{\lambda_1}$ the results will be expressed in terms of $\cot\theta$; $\cot\theta$ was selected rather than $\tan\theta$ to make the denominator as simple as possible.

$$\cot\theta = \left[\left(x_2^2 - \frac{k}{x_3}\right)\Sigma k_i \left(\frac{F}{m}\right)^i + x_1^2\ \Sigma \ell_i \left(\frac{F}{m}\right)^i + x_1 x_3 \Sigma r_i \left(\frac{F}{m}\right)^i + x_3^2\ \Sigma s_i \left(\frac{F}{m}\right)^i + c_1\left(x_2^2 - \frac{k}{x_3}\right) t + c_2 x_1 x_3 t + c_3 x_1 x_2 t + c_4 x_1 x_2 + c_5 x_2 x_3\right] \Bigg/ \left[x_1 x_2 \Sigma g_i \left(\frac{F}{m}\right)^i + x_2 x_3 \Sigma h_i \left(\frac{F}{m}\right)^i + c_6 x_1 x_2 t + c_7 x_2 x_3 t + c_8 x_2^2 t + c_9 x_2^2\right] \qquad (27)$$

In recent guidance polynomials the series have been truncated after second order terms involving $\frac{F}{m}$. If we assume that all terms involving $\frac{F}{m}$ of the rational function are dropped off after $\left(\frac{F}{m}\right)^2$, equation (27) becomes

$$\cot\theta = \left[a_1 x_1^2 + a_3 x_1 x_2 + a_4 x_2 x_3 + a_5 x_1 x_3 + a_6\left(x_2^2 - \frac{k}{x_3}\right) + a_2 x_3^2 + a_7 x_1 x_2 t + a_8 x_1 x_3 t + a_9\left(x_2^2 - \frac{k}{x_3}\right) t + a_{10}\, x_1^2 \left(\frac{F}{m}\right) + a_{11} x_1 x_3 \left(\frac{F}{m}\right) + a_{12}\, x_3^2 \left(\frac{F}{m}\right) + a_{13}\left(x_2^2 - \frac{k}{x_3}\right)\left(\frac{F}{m}\right) + a_{14} x_1^2 \left(\frac{F}{m}\right)^2 + a_{15}\, x_1 x_3 \left(\frac{F}{m}\right)^2 + a_{16} x_3^2 \left(\frac{F}{m}\right)^2 + a_{17}\left(x_2^2 - \frac{k}{x_3}\right)\left(\frac{F}{m}\right)^2\right] \Bigg/ \left[a_{18} x_1 x_2 + a_{19} x_2 x_3 + a_{20}\, x_2^2 + a_{21} x_1 x_2 t + a_{22}\, x_2 x_3 t + a_{23} x_2^2 t + a_{24} x_1 x_2 \left(\frac{F}{m}\right) + a_{25}\, x_2 x_3 \left(\frac{F}{m}\right) + a_{26} x_1 x_2 \left(\frac{F}{m}\right)^2 \; a_{27}\, x_2 x_3 \left(\frac{F}{m}\right)\right] \qquad (28)$$

Of course the assumption that the series involving $\frac{F}{m}$ can be truncated after second order terms may or may not be correct. The order of the terms at which all other terms would be insignificant would need to be determined by considering problems on an electronic computer.

Thus the cot θ is expressed in terms of a rational function of state variables involving 17 terms in the numerator and 10 terms in the denominator. To use this function as a guidance function would require that 29 constants be found by some curve fitting technique such that the function would give a good representation of the space of optimal trajectories.

It is of interest to investigate what additional assumptions will need to be made in order to change our rational function to a polynomial. This can be accomplished in either of the following ways.

In reference (1) Mr. Moyer considered the equations (4) without the condition that $g_4 = \dot{x}_4 - \frac{x_2}{x_3} = 0$. If we take the assumption of Mr. Moyer we will be able to obtain a polynomial.

If we are not able to make the above assumption then we can still obtain a polynomial by certain assumptions relative to equation (15). In this equation group the $\dot{x}_4 \lambda_4$ (which can be written as $c \frac{x_2}{x_3}$) on the right side of the equation to obtain:

$$\frac{x_2^2}{x_3} \lambda_1 - \frac{k}{x_3^2} \lambda_1 - \frac{x_1 x_2}{x_3} \lambda_2 + x_1 \lambda_3 = c_1 - \frac{c x_2}{x_3} + \dot{m}\lambda_5 + m\dot{\lambda}_5 ,$$

Now assume that $c_1 \frac{c_2 x_2}{x_3} + \dot{m}\lambda_5 + m\dot{\lambda}_5$ can be written as a power series involving $\frac{F}{m}$: Under this assumption , then equation (27) may be simplified in the following manner. Multiply the numerator and denominator by $\frac{F}{m}$; divide by $x_2\ x_3$; and then divide the numerator and denominator by the power series $\Sigma h_i \left(\frac{F}{m}\right)^i$ to obtain:

$$\cot\theta = \frac{1}{\dot\phi}\left[\frac{(\dot\phi^2 - \frac{k}{r^3}) \sum\limits_{i=0} a_i \left(\frac{F}{m}\right)^i + \left(\frac{\dot r}{r}\right)^2 \sum\limits_{i=1} b_i \left(\frac{F}{m}\right)^i + \frac{\dot r}{r} \sum\limits_{i=0} c_i \left(\frac{F}{m}\right)^i + \sum\limits_{i=1} d_i \left(\frac{F}{m}\right)^i}{1 + \left(\frac{\dot r}{r}\right) \sum\limits_{i=0} e_i \left(\frac{F}{m}\right)^i}\right] \tag{29}$$

or

$$\cot\theta = \frac{1}{\dot\phi}\left[\frac{\frac{r}{\dot r}(\dot\phi^2 - \frac{k}{r^3}) \sum\limits_{i=0} f_i \left(\frac{F}{m}\right)^i + \frac{\dot r}{r} \sum\limits_{i=1} g_i \left(\frac{F}{m}\right)^i + \sum\limits_{i=0} h_i \left(\frac{F}{m}\right)^i + \frac{r}{\dot r} \sum\limits_{i=1} j_i \left(\frac{F}{m}\right)^i}{1 + \frac{r}{\dot r} \Sigma k_i \left(\frac{F}{m}\right)^i}\right] \tag{30}$$

These two expressions for $\cot\theta$ are equivalent. The terms in the denominators are reciprocals. Now if $\frac{\dot r}{r} \sum\limits_{i=0} e_i \left(\frac{F}{m}\right)^i$ converges to a number in absolute value less than 1, the denominator can be expanded as a negative binomial in the numerator. If $\frac{\dot r}{r} \Sigma e_i \left(\frac{F}{m}\right)^i$ is greater than one then $\frac{r}{\dot r} \Sigma k_i \left(\frac{F}{m}\right)^i$ is less than one and the second denominator can be expanded as a negative binomial in the numerator. Thus we have two polynomials representing $\cot\theta$ depending on whether $\frac{\dot r}{r} \sum\limits_{i=0} e_i \left(\frac{F}{m}\right)^i$ is less than or greater than one. Since in general the size of $\frac{\dot r}{r} \sum\limits_{i=0} e_i \left(\frac{F}{m}\right)^i$ is not known, we write down all the terms contained in both polynomials. Notice that the model obtained involves r, $\dot r$ and $\dot\phi$ and could possibly reduce the number of terms needed in a polynomial guidance function.

$$
\begin{aligned}
\cot\theta = \frac{1}{\dot{\phi}}\Bigg[& a_1 + a_2 \frac{F}{m} + a_3\left(\frac{F}{m}\right)^2 + a_4\left(\frac{r}{\dot{r}}\right) + a_5\left(\frac{r}{\dot{r}}\right)\left(\frac{F}{m}\right) \\
& + a_6\left(\frac{r}{\dot{r}}\right)\left(\frac{F}{m}\right)^2 + a_7\left(\frac{r}{\dot{r}}\right)^2 + a_8\left(\frac{r}{\dot{r}}\right)^2\left(\frac{F}{m}\right) + a_9\left(\frac{r}{\dot{r}}\right)^2\left(\frac{F}{m}\right)^2 \\
& + a_{10}\left(\frac{\dot{r}}{r}\right) + a_{11}\left(\frac{\dot{r}}{r}\right)\left(\frac{F}{m}\right) + a_{12}\left(\frac{\dot{r}}{r}\right)\left(\frac{F}{m}\right)^2 + a_{13}\left(\frac{\dot{r}}{r}\right)^2 \\
& + a_{14}\left(\frac{\dot{r}}{r}\right)^2\left(\frac{F}{m}\right) + a_{15}\left(\frac{\dot{r}}{r}\right)^2\left(\frac{F}{m}\right)^2 \\
& + a_{16}\left(\dot{\phi}^2 - \frac{k}{r^3}\right) + a_{17}\left(\dot{\phi}^2 - \frac{k}{r^3}\right)\left(\frac{F}{m}\right) + a_{18}\left(\dot{\phi}^2 - \frac{k}{r^3}\right)\left(\frac{F}{m}\right)^2 \\
& + a_{19}\left(\dot{\phi}^2 - \frac{k}{r^3}\right)\left(\frac{r}{\dot{r}}\right) + a_{20}\left(\dot{\phi}^2 - \frac{k}{r^3}\right)\left(\frac{r}{\dot{r}}\right)\left(\frac{F}{m}\right) + a_{21}\left(\dot{\phi}^2 - \frac{k}{r^3}\right)\left(\frac{r}{\dot{r}}\right)\left(\frac{F}{m}\right)^2 \\
& + a_{22}\left(\dot{\phi}^2 - \frac{k}{r^3}\right)\left(\frac{r}{\dot{r}}\right)^2 + a_{23}\left(\dot{\phi}^2 - \frac{k}{r^3}\right)\left(\frac{r}{\dot{r}}\right)^2\left(\frac{F}{m}\right) + \cdots \Bigg] \qquad (31)
\end{aligned}
$$

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A Method for Deriving A Function To Satisfy A Given Least Squares Error Tolerance

by

Daniel E. Dupree, James O'Neil, Edward Anders

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SUMMARY

From a given vector derived previously, an ideal function is developed which satisfies a specified least squares error tolerance.

THE FUNCTION

In the vector $\bar{\varphi}_{N+1} = (\lambda_0, \lambda_1, \ldots, \lambda_n)$, computed in [3], suppose we let λ_i be the value of some ideal function $\varphi_{N+1}(\beta)$ at β_i; i.e., $\varphi_{N+1}(\beta_i) = \lambda_i$. Then this ideal function assures us that the error E, where

$$E = \sum_{i=0}^{n} \left[X(\beta_i) - \sum_{j=0}^{N} A_j \varphi_j(\beta_i) - A_{N+1}\varphi_{N+1}(\beta_i) \right]^2,$$

is less than the imposed tolerance δ. Since we know the values of this ideal function at the tabular values β_i, our next objective is to develop a technique for computing $\varphi_{N+1}(\beta')$, for some value $\beta' \neq \beta_i$, $i = 0,1,\dots,n$, such that the error obtained by using $\sum_{j=0}^{N+1} A_j\varphi_j(\beta')$ to approximate $X(\beta')$, in the sense of least squares, is as small, if not smaller, than the error obtained by approximating $X(\beta')$ with $\sum_{j=0}^{N} A_j\varphi_j(\beta')$. We obtain this value $\varphi_{N+1}(\beta')$ in the following manner.

First, we compute $A_{N+1}(k)$, $k = -1, 0, 1, \dots, N$, $\bar{e}_{N+1}$ and A'_{N+1} as follows:

$$A_{N+1}(-1) = \frac{1}{\left\| \bar{\varphi}_{N+1} - \sum_{j=0}^{N} (\bar{\varphi}_{N+1}, \bar{e}_j)\, \bar{e}_j \right\|}$$

$$A_{N+1}(0) = A_{N+1}(-1)\, A_0(-1)(\bar{\varphi}_{N+1}, \bar{\varphi}_0)$$

$$\vdots$$

$$A_{N+1}(N) = A_{N+1}(-1)\, A_N(-1)(\bar{\varphi}_{N+1}, \bar{\varphi}_N) - \sum_{j=0}^{N-1} A_{N+1}(j)\, A_N(j).$$

$$\bar{e}_{N+1} = A_{N+1}(-1)\, \bar{\varphi}_{N+1} - \sum_{j=0}^{N} A_{N+1}(j)\, \bar{e}_j$$

$$A'_{N+1} = (\bar{X}, \bar{e}_{N+1}).$$

Finally, compute the (N+2) A_j's, $j = 0, 1, \dots, N+1$, as follows:

$$A_{N+1} = A'_{N+1}\, A_{N+1}(-1)$$

$$A_N = A_N(-1)\left[A'_N - A'_{N+1}\, A_{N+1}(N)\right]$$

$$A_{N-1} = A_{N-1}(-1)\left\{A'_{N-1} - A'_N\, A_N(N-1) + A'_{N+1}\left[-A_{N+1}(N-1) + A_{N+1}(N)A_N(N-1)\right]\right\},$$

$$\vdots$$

Now let $\beta_{i'}$ be a β_i such that $\| \beta_{i'} - \beta' \| = \min\limits_{0 \le i \le n} \left\{ \| \beta_i - \beta' \| \right\}$

and let us define the following function:

$$\sum_{j=0}^{N} A_j \varphi_j(\beta') + A_{N+1}\, M(\beta'),$$

$$\text{where } M(\beta') = \lambda_{i'} \left[\frac{L(\beta_{i'}) - 2\,\| \beta_{i'} - \beta' \|}{L(\beta_{i'})}\right],$$

$$\text{for } 2\,\| \beta_{i'} - \beta' \| < L(\beta_{i'}),$$

$$= 0, \text{ otherwise},$$

$$\text{where } L(\beta_{i'}) = \min_{\substack{0 \le i \le n \\ i \ne i'}} \left\{ \| \beta_i - \beta_{i'} \| \right\}.$$

Thus, when β' is chosen, we are able to use the function above to approximate $X(\beta')$, being assured that the approximation obtained here is no worse than the value $\sum_{j=0}^{N} A_j \varphi_j(\beta')$ obtained by using the initial least squares approximating function.

Writing this multiple of λ_i as

$$\frac{\frac{1}{2} L(\beta_{i'}) - \| \beta_{i'} - \beta' \|}{\frac{1}{2} L(\beta_{i'})}$$

we see that we have a factor which varies from zero to one as β' varies from a position on the boundary to a position at the center of the ball

$$\left\{ \beta \;\middle|\; \| \beta_{i'} - \beta' \| \leq \frac{1}{2} L(\beta_{i'}) \right\} .$$

Thus, the factor λ_i, which was derived in association with the vector $\beta_{i'}$ is weighted depending on the nearness of β' to $\beta_{i'}$.

For a particular β', we may have a possibility of multiple choices for $\beta_{i'}$. Perhaps, more than one of these would satisfy

$$2 \| \beta_{i'} - \beta' \| < L(\beta_{i'}).$$

This situation depends on the configuration of the β_i's and on the orientation of β' with the β_i's near it in the norm sense.

Suppose there are m choices of $\beta_{i'}$, and r of them satisfy $2 \| \beta_{i'} - \beta' \| < L(\beta_{i'})$. Let $\beta_{i'_1}, \beta_{i'_2}, \ldots, \beta_{i'_r}$ denote these $\beta_{i'}$'s.

Let $H_{i'_{t_o}} = \max\limits_{1 \leq t \leq r} \left[\dfrac{L(\beta_{i'_t}) - 2 \| \beta_{i'} - \beta' \|}{L(\beta_{i'_t})} \right]$ and $M(\beta') =$

$\lambda_{i'_{t_o}} \cdot H_{i'_{t_o}}$. Thus we orient β' with the β_i which exerts the most influence on β'.

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on Studies in the Fields of

SPACE FLIGHT AND GUIDANCE THEORY

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of Marshall Space Flight Center

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